

## Algebra

- [1] Given any set  $A = \{a_1, a_2, a_3, a_4\}$  of four distinct positive integers, we denote the sum  $a_1 + a_2 + a_3 + a_4$  by  $s_A$ . Let  $n_A$  denote the number of pairs  $(i, j)$  with  $1 \leq i < j \leq 4$  for which  $a_i + a_j$  divides  $s_A$ . Find all sets  $A$  of four distinct positive integers which achieve the largest possible value of  $n_A$ .

*Proposed by Fernando Campos, Mexico*

- [2] Determine all sequences  $\{x_1, x_2, \dots, x_{2011}\}$  of positive integers, such that for every positive integer  $n$  there is an integer  $a$  with  $\sum_{j=1}^{2011} j \cdot x_j^n = a^{n+1} + 1$ .
- [3] Determine all pairs  $(f, g)$  of functions from the set of real numbers to itselfs that satisfy

$$g(f(x+y)) = f(x) + (2x+y)g(y)$$

for all real numbers  $x$  and  $y$ .

- [4] Determine all pairs  $(f, g)$  of functions from the set of positive integers to itselfs that satisfy

$$f^{g(n)+1}(n) + g^{f(n)}(n) = f(n+1) - g(n+1) + 1$$

for every positive integer  $n$ . Here,  $f^k(n)$  means  $\underbrace{f(f(\dots f(n)\dots))}_k$ .

- [5] Prove that for every positive integer  $n$ , the set  $\{2, 3, 4, \dots, 3n+1\}$  can be partitioned into  $n$  triples in such a way that the numbers from each triple are the lengths of the sides of some obtuse triangle.
- [6] Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a real-valued function defined on the set of real numbers that satisfies

$$f(x+y) \leq yf(x) + f(f(x))$$

for all real numbers  $x$  and  $y$ . Prove that  $f(x) = 0$  for all  $x \leq 0$ .

*Proposed by Igor Voronovich, Belarus*

- [7] Let  $a, b$  and  $c$  be positive real numbers satisfying  $\min(a+b, b+c, c+a) > \sqrt{2}$  and  $a^2 + b^2 + c^2 = 3$ . Prove that

$$\frac{a}{(b+c-a)^2} + \frac{b}{(c+a-b)^2} + \frac{c}{(a+b-c)^2} \geq \frac{3}{(abc)^2}.$$

## Combinatorics

- [1] Let  $n > 0$  be an integer. We are given a balance and  $n$  weights of weight  $2^0, 2^1, \dots, 2^{n-1}$ . We are to place each of the  $n$  weights on the balance, one after another, in such a way that the right pan is never heavier than the left pan. At each step we choose one of the weights that has not yet been placed on the balance, and place it on either the left pan or the right pan, until all of the weights have been placed. Determine the number of ways in which this can be done.

*Proposed by Morteza Saghafian, Iran*

- [2] Suppose that 1000 students are standing in a circle. Prove that there exists an integer  $k$  with  $100 \leq k \leq 300$  such that in this circle there exists a contiguous group of  $2k$  students, for which the first half contains the same number of girls as the second half.
- [3] Let  $\mathcal{S}$  be a finite set of at least two points in the plane. Assume that no three points of  $\mathcal{S}$  are collinear. A *windmill* is a process that starts with a line  $\ell$  going through a single point  $P \in \mathcal{S}$ . The line rotates clockwise about the *pivot*  $P$  until the first time that the line meets some other point belonging to  $\mathcal{S}$ . This point,  $Q$ , takes over as the new pivot, and the line now rotates clockwise about  $Q$ , until it next meets a point of  $\mathcal{S}$ . This process continues indefinitely. Show that we can choose a point  $P$  in  $\mathcal{S}$  and a line  $\ell$  going through  $P$  such that the resulting windmill uses each point of  $\mathcal{S}$  as a pivot infinitely many times.

*Proposed by Geoffrey Smith, United Kingdom*

- [4] Determine the greatest positive integer  $k$  that satisfies the following property: The set of positive integers can be partitioned into  $k$  subsets  $A_1, A_2, \dots, A_k$  such that for all integers  $n \geq 15$  and all  $i \in \{1, 2, \dots, k\}$  there exist two distinct elements of  $A_i$  whose sum is  $n$ .
- [5] Let  $m$  be a positive integer, and consider a  $m \times m$  checkerboard consisting of unit squares. At the centre of some of these unit squares there is an ant. At time 0, each ant starts moving with speed 1 parallel to some edge of the checkerboard. When two ants moving in the opposite directions meet, they both turn  $90^\circ$  clockwise and continue moving with speed 1. When more than 2 ants meet, or when two ants moving in perpendicular directions meet, the ants continue moving in the same direction as before they met. When an ant reaches one of the edges of the checkerboard, it falls off and will not re-appear.

Considering all possible starting positions, determine the latest possible moment at which the last ant falls off the checkerboard, or prove that such a moment does not necessarily exist.

- [6] Let  $n$  be a positive integer, and let  $W = \dots x_{-1}x_0x_1x_2\dots$  be an infinite periodic word, consisting of just letters  $a$  and/or  $b$ . Suppose that the minimal period  $N$  of  $W$  is greater than  $2^n$ .

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A finite nonempty word  $U$  is said to *appear* in  $W$  if there exist indices  $k \leq \ell$  such that  $U = x_k x_{k+1} \dots x_\ell$ . A finite word  $U$  is called *ubiquitous* if the four words  $Ua$ ,  $Ub$ ,  $aU$ , and  $bU$  all appear in  $W$ . Prove that there are at least  $n$  ubiquitous finite nonempty words.

- [7] On a square table of 2011 by 2011 cells we place a finite number of napkins that each cover a square of 52 by 52 cells. In each cell we write the number of napkins covering it, and we record the maximal number  $k$  of cells that all contain the same nonzero number. Considering all possible napkin configurations, what is the largest value of  $k$ ?

## Geometry

- [1] Let  $ABC$  be an acute triangle. Let  $\omega$  be a circle whose centre  $L$  lies on the side  $BC$ . Suppose that  $\omega$  is tangent to  $AB$  at  $B'$  and  $AC$  at  $C'$ . Suppose also that the circumcentre  $O$  of triangle  $ABC$  lies on the shorter arc  $B'C'$  of  $\omega$ . Prove that the circumcircle of  $ABC$  and  $\omega$  meet at two points.
  
- [2] Let  $A_1A_2A_3A_4$  be a non-cyclic quadrilateral. Let  $O_1$  and  $r_1$  be the circumcentre and the circumradius of the triangle  $A_2A_3A_4$ . Define  $O_2, O_3, O_4$  and  $r_2, r_3, r_4$  in a similar way. Prove that
 
$$\frac{1}{O_1A_1^2 - r_1^2} + \frac{1}{O_2A_2^2 - r_2^2} + \frac{1}{O_3A_3^2 - r_3^2} + \frac{1}{O_4A_4^2 - r_4^2} = 0.$$
  
- [3] Let  $ABCD$  be a convex quadrilateral whose sides  $AD$  and  $BC$  are not parallel. Suppose that the circles with diameters  $AB$  and  $CD$  meet at points  $E$  and  $F$  inside the quadrilateral. Let  $\omega_E$  be the circle through the feet of the perpendiculars from  $E$  to the lines  $AB, BC$  and  $CD$ . Let  $\omega_F$  be the circle through the feet of the perpendiculars from  $F$  to the lines  $CD, DA$  and  $AB$ . Prove that the midpoint of the segment  $EF$  lies on the line through  $\omega_E$  and  $\omega_F$ .
  
- [4] Let  $ABC$  be an acute triangle with circumcircle  $\Omega$ . Let  $B_0$  be the midpoint of  $AC$  and let  $C_0$  be the midpoint of  $AB$ . Let  $D$  be the foot of the altitude from  $A$  and let  $G$  be the centroid of the triangle  $ABC$ . Let  $\omega$  be a circle through  $B_0$  and  $C_0$  that is tangent to the circle  $\Omega$  at a point  $X \neq A$ . Prove that the points  $D, G$  and  $X$  are collinear.
  
- [5] Let  $ABC$  be a triangle with incentre  $I$  and circumcircle  $\omega$ . Let  $D$  and  $E$  be the second intersection points of  $\omega$  with  $AI$  and  $BI$ , respectively. The chord  $DE$  meets  $AC$  at a point  $F$ , and  $BC$  at a point  $G$ . Let  $P$  be the intersection point of the line through  $F$  parallel to  $AD$  and the line through  $G$  parallel to  $BE$ . Suppose that the tangents to  $\omega$  at  $A$  and  $B$  meet at a point  $K$ . Prove that the three lines  $AE, BD$  and  $KP$  are either parallel or concurrent.
  
- [6] Let  $ABC$  be a triangle with  $AB = AC$  and let  $D$  be the midpoint of  $AC$ . The angle bisector of  $\angle BAC$  intersects the circle through  $D, B$  and  $C$  at the point  $E$  inside the triangle  $ABC$ . The line  $BD$  intersects the circle through  $A, E$  and  $B$  in two points  $B$  and  $F$ . The lines  $AF$  and  $BE$  meet at a point  $I$ , and the lines  $CI$  and  $BD$  meet at a point  $K$ . Show that  $I$  is the incentre of triangle  $KAB$ .
  
- [7] Let  $ABCDEF$  be a convex hexagon all of whose sides are tangent to a circle  $\omega$  with centre  $O$ . Suppose that the circumcircle of triangle  $ACE$  is concentric with  $\omega$ . Let  $J$  be the foot of the perpendicular from  $B$  to  $DF$  intersects the line  $EO$  at a point  $K$ . Let  $L$  be the foot of the perpendicular from  $K$  to  $DE$ . Prove that  $DJ = DL$ .

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- 8 Let  $ABC$  be an acute triangle with circumcircle  $\Gamma$ . Let  $\ell$  be a tangent line to  $\Gamma$ , and let  $\ell_a, \ell_b$  and  $\ell_c$  be the lines obtained by reflecting  $\ell$  in the lines  $BC, CA$  and  $AB$ , respectively. Show that the circumcircle of the triangle determined by the lines  $\ell_a, \ell_b$  and  $\ell_c$  is tangent to the circle  $\Gamma$ .

*Proposed by Japan*

## Number Theory

- [1] For any integer  $d > 0$ , let  $f(d)$  be the smallest possible integer that has exactly  $d$  positive divisors (so for example we have  $f(1) = 1$ ,  $f(5) = 16$ , and  $f(6) = 12$ ). Prove that for every integer  $k \geq 0$  the number  $f(2^k)$  divides  $f(2^{k+1})$ .
- [2] Consider a polynomial  $P(x) = \prod_{j=1}^9 (x + d_j)$ , where  $d_1, d_2, \dots, d_9$  are nine distinct integers. Prove that there exists an integer  $N$ , such that for all integers  $x \geq N$  the number  $P(x)$  is divisible by a prime number greater than 20.
- [3] Let  $n \geq 1$  be an odd integer. Determine all functions  $f$  from the set of integers to itself, such that for all integers  $x$  and  $y$  the difference  $f(x) - f(y)$  divides  $x^n - y^n$ .
- [4] For each positive integer  $k$ , let  $t(k)$  be the largest odd divisor of  $k$ . Determine all positive integers  $a$  for which there exists a positive integer  $n$ , such that all the differences

$$t(n+a) - t(n); t(n+a+1) - t(n+1), \dots, t(n+2a-1) - t(n+a-1)$$

are divisible by 4.

- [5] Let  $f$  be a function from the set of integers to the set of positive integers. Suppose that, for any two integers  $m$  and  $n$ , the difference  $f(m) - f(n)$  is divisible by  $f(m-n)$ . Prove that, for all integers  $m$  and  $n$  with  $f(m) \leq f(n)$ , the number  $f(n)$  is divisible by  $f(m)$ .

*Proposed by Mahyar Sefidgaran, Iran*

- [6] Let  $P(x)$  and  $Q(x)$  be two polynomials with integer coefficients, such that no nonconstant polynomial with rational coefficients divides both  $P(x)$  and  $Q(x)$ . Suppose that for every positive integer  $n$  the integers  $P(n)$  and  $Q(n)$  are positive, and  $2^{Q(n)} - 1$  divides  $3^{P(n)} - 1$ . Prove that  $Q(x)$  is a constant polynomial.
- [7] Let  $p$  be an odd prime number. For every integer  $a$ , define the number  $S_a = \sum_{j=1}^{p-1} \frac{a^j}{j}$ . Let  $m, n \in \mathbb{Z}$ , such that  $S_3 + S_4 - 3S_2 = \frac{m}{n}$ . Prove that  $p$  divides  $m$ .
- [8] Let  $k \in \mathbb{Z}^+$  and set  $n = 2^k + 1$ . Prove that  $n$  is a prime number if and only if the following holds: there is a permutation  $a_1, \dots, a_{n-1}$  of the numbers  $1, 2, \dots, n-1$  and a sequence of integers  $g_1, \dots, g_{n-1}$ , such that  $n$  divides  $g_i^{a_i} - a_{i+1}$  for every  $i \in \{1, 2, \dots, n-1\}$ , where we set  $a_n = a_1$ .