Problems in Algebra

Section 2 Polynomial Equations

For a polynomial equation
$$P(x) = c_0x^n + c_1x^{n-1} + \cdots + c_n = 0,$$
where $c_0, c_1, \ldots, c_n$ are complex numbers, we know from the Fundamental Theorem of Algebra that there are complex numbers x that satisfy the equation. A *root* of the equation $P(x) = 0$ is also referred to as a *zero* of the polynomial P. To solve the equation means to specify all of the roots of the equation exactly by a finite number of standard operations ($+, -, \times, \div, \sqrt[n]{\ }$) applied to the coefficients.

Consider the quadratic equation $ax^2 + bx + c = 0$. Completing the square, one finds the complete solution set,
$$S = \left\{ \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \right\}.$$

Such formulas also exist for cubic and quartic equations. But, as Abel first proved, arbitrary equations of degree five and higher cannot be solved in such away.

The following plan of attack is suggested:

- Rational roots?
- Special form?
- If all else fails ...

Recall the **Rational Root Theorem** *If $c_0, c_1, \ldots, c_n$ are integers and $x = a/b$ is a rational root of the equation*
$$P(x) = c_0x^n + c_1x^{n-1} + \cdots + c_n = 0,$$
then $a|c_n$ and $b|c_0$.

Example 12 *Solve the equation* $3x^3 - 7x^2 + 17x - 5 = 0$.

Solution We first look for rational solutions. From the Rational Root Theorem, if $x = a/b$ is a root then $a|5$ and $b|3$. Trying the possible combinations, we find that $x = 1/3$ is in fact a root. Now we can factor:
$$(3x - 1)(x^2 - 2x + 5) = 0.$$
Thus the solution set is
$$S = \{1/3, 1 \pm 2i\}.$$
■

An equation of even degree
$$c_0x^{2n} + c_1x^{2n-1} + \cdots + c_{2n} = 0$$
is called a *reciprocal equation* if $c_k = c_{2n-k}$ for all k. In other words, the sequence of coefficients reads the same from right to left as it does from left to right. For such an equation, the transformation $z = x + x^{-1}$ reduces the problem to that of solving a polynomial equation of degree n (half the original degree).

Example 13 (1973 IMO) *Let a and b be real numbers for which that equation*
$$x^4 + ax^3 + bx^2 + ax + 1 = 0$$
has at least one real solution. For all such pairs (a, b), find the minimum value of $a^2 + b^2$.

Solution Putting $y = x + 1/x$, the equation becomes $y^2 + ay + b - 2 = 0$. We have

$$y = \frac{-a \pm \sqrt{a^2 + 8 - 2b}}{2}.$$

If the original equation has a real root, then $|y| \geq 2$ and hence $|a| + \sqrt{a^2 + 8 - 2b} \geq 4$. Simplifying, we get $2|a| - b \geq 2$, and

$$a^2 + b^2 \geq a^2 + (2 - 2|a|)^2 = 5(|a| - 4/5)^2 + 4/5.$$

The least possible value of $a^2 + b^2$ is $4/5$, achieved when $a = 4/5$, $b = -2/5$. In this case, the original equation is

$$x^4 + 4/5\,x^3 - 2/5\,x^2 + 4/5\,x + 1 = (x+1)^2(x^2 - 6/5\,x + 1).$$ ■

There are many other cases where the equation is special and can be solved easily if one makes the right observation.

Example 14 *Solve* $x^4 + 2x^3 + 7x^2 + 6x + 8 = 0$.

Solution Observe that the equation can be written as $(x^2 + x)^2 + 6(x^2 + x) + 8 = 0$. Thus $(x^2 + x + 4)(x^2 + x + 2) = 0$, and the complete solution set is

$$S = \left\{ \frac{-1 \pm i\sqrt{15}}{2},\ \frac{-1 \pm i\sqrt{7}}{2} \right\}.$$ ■

Exercise 12 (M& IQ 3) *Solve the equation* $(x+1)(x+2)(x+3)(x+4) = -1$.

Exercise 13 (1993 BO) *Given the equation*

$$(x^2 - 3x - 2)^2 - 3(x^2 - 3x - 2) - 2 - x = 0.$$

Prove that the roots of the equation $x^2 - 4x - 2 = 0$ *are roots of the initial equation and find all roots of the given equation.*

Exercise 14 (1991 MAΘ) *The equation with roots* $3 + \sqrt{2}, 3 - \sqrt{2}, -3 + i\sqrt{2}$ *and* $-3 - i\sqrt{2}$ *is in the form* $x^4 + Ax^3 + Bx^2 + Cx + D = 0$. *Find* $A + B + C + D$.

Exercise 15 *Prove that*

$$a^2 + ab + b^2 \geq 3(a + b - 1)$$

for all real numbers a, b.

Although there is no general formula that takes us from the coefficients $c_0, c_1, \ldots, c_n$ of the polynomial equation

$$c_0 x^n + c_1 x^{n-1} + \cdots + c_n = 0$$

to its roots $x_1, x_2, \ldots, x_n$, there are formulas that take us from $c_0, c_1, \ldots, c_n$ to a large and important class of functions of the roots. These are the *symmetric functions*. A symmetric function of $x_1, x_2, \ldots, x_n$ is one whose value is unchanged if $x_1, x_2, \ldots, x_n$ are permuted arbitrarily.

For example, each of the following is a symmetric function of three variables:

$$\begin{aligned} P(x_1, x_2, x_3) &= x_1x_2 + x_2x_3 + x_3x_1, \\ Q(x_1, x_2, x_3) &= x_1^3 + x_2^3 + x_3^3, \\ R(x_1, x_2, x_3) &= \tfrac{x_2+x_3}{x_1} + \tfrac{x_3+x_1}{x_2} + \tfrac{x_1+x_2}{x_3}. \end{aligned}$$

Certain symmetric functions serve as building blocks for all the rest. Let

$$\sigma_k = \sum x_{i_1} x_{i_2} \cdots x_{i_k},$$

where the sum is taken over all $\binom{n}{k}$ choices of the indices $i_1, i_2, \ldots, i_k$ from $\{1, 2, \ldots, n\}$. Then σ_k is called the kth *elementary symmetric function* of $x_1, x_2, \ldots, x_n$.

Symmetric Function Theorem *Every symmetric polynomial function of $x_1, x_2, \ldots, x_n$ is a polynomial function of $\sigma_1, \sigma_2, \ldots, \sigma_n$.*
The same conclusion holds if "polynomial" is replaced by "rational function."

As an illustration, for $n = 3$ the elementary symmetric functions are

$$\begin{aligned}
\sigma_1 &= x_1 + x_2 + x_3, \\
\sigma_2 &= x_1x_2 + x_2x_3 + x_3x_1, \\
\sigma_3 &= x_1x_2x_3,
\end{aligned}$$

and it is easy to check that the examples given earlier can be expressed in terms of these as follows:

$$\begin{aligned}
x_1x_2 + x_2x_3 + x_3x_1 &= \sigma_1 \\
x_1^3 + x_2^3 + x_3^3 &= \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3 \\
\tfrac{x_2+x_3}{x_1} + \tfrac{x_3+x_1}{x_2} + \tfrac{x_1+x_2}{x_3} &= \tfrac{\sigma_1\sigma_2 - 3\sigma_3}{\sigma_3}.
\end{aligned}$$

Another important fact involves the relationship between the coefficients of a polynomial and the elementary symmetric functions of its zeros.

Theorem *Let $x_1, x_2, \ldots, x_n$ be the roots of the polynomial equation $x^n + c_1x^{n-1} + \cdots + c_n = 0$, and let σ_k be the kth elementary symmetric function of the x_i. Then $\sigma_k = (-1)^k c_k$, $k = 1, 2, \ldots, n$.*

Example 15 (1977 USAMO) *If a and b are two roots of $x^4 + x^3 - 1 = 0$, prove that ab is a root of $x^6 + x^4 + x^3 - x^2 - 1 = 0$.*

Solution Suppose the roots of $x^4 + x^3 - 1 = 0$ are a, b, c, d. Then

$$\begin{aligned}
a + b + c + d &= -1, \\
ab + ac + ad + bc + bd + cd &= 0, \\
abc + bcd + cda + dab &= 0, \\
abcd &= -1.
\end{aligned}$$

The equations can be rewritten as

$$\begin{aligned}
(a + b) + (c + d) &= -1, \\
ab + (a + b)(c + d) + cd &= 0, \\
ab(c + d) + (a + b)cd &= 0, \\
ab \cdot cd &= -1.
\end{aligned}$$

From the first and the last equations, we get $(c + d) = -1 - (a + b)$ and $cd = -1/(ab)$. Putting into the second and the third equations,

$$\begin{aligned}
ab + (a + b)(-1 - (a + b)) - 1/ab &= 0, \\
ab(-1 - (a + b)) - (a + b)/ab &= 0.
\end{aligned}$$

Hence $(a+b) = -\frac{(ab)^2}{1+(ab)^2}$, and

$$ab - \left[-\frac{(ab)^2}{1+(ab)^2} \cdot \left(1 - \frac{(ab)^2}{1+(ab)^2}\right)\right] - \frac{1}{ab} = 0.$$

The result follows by simplifying the above equation. ■

Example 16 (1984 USAMO) *The product of two of the four roots of the quartic equation $x^4 - 18x^3 + kx^2 + 200x - 1984 = 0$ is -32. Determine the value of k.*

Solution Suppose the two roots satisfy the quadratic equation $x^2 + ax - 32 = 0$. The other two roots have product $(-1984)/(-32) = 62$, and they satisfy a quadratic equation $x^2 + bx + 62 = 0$. Hence

$$\begin{aligned}
& x^4 - 18x^3 + kx^2 + 200x - 1984 \\
= \; & (x^2 + ax - 32)(x^2 + bx + 62) \\
= \; & x^4 + (a+b)x^3 + (ab+30)x^2 + (62a - 32b)x - 1984 \\
= \; & 0.
\end{aligned}$$

Equating coefficients, we get $a+b = -18$, and $62a - 32b = 200$. Solving the simultaneous equations, $a = -4$, $b = -14$. Hence $k = (-4)(-14) + 30 = 86$. ■

Exercise 16 (1991 MAΘ) *The equation $x^4 - 16x^3 + 94x^2 + px + q = 0$ has two double roots. Find $p+q$.*

Exercise 17 Let a, b, c be real numbers such that $a + b + c > 0$, $bc + ca + ab > 0$, $abc > 0$. Prove that a, b, c are all positive.

Exercise 18 (1981 AHSME) *If a, b, c, d are the solutions of the equation $x^4 - bx - 3 = 0$, find the polynomial with leading coefficient 3 whose roots are*

$$\frac{a+b+c}{d^2}, \quad \frac{a+b+d}{c^2}, \quad \frac{a+c+d}{b^2}, \quad \text{and} \quad \frac{b+c+d}{a^2}.$$

Exercise 19 *The polynomial $ax^3 + bx^2 + cx + d$ has integer coefficients a, b, c, d with ad odd and bc even. Show that at least one zero of the polynomial is irrational.*

Theorem (Newton's Formulas for Power Sums) *Let*

$$S_p = x_1^p + x_2^p + \cdots + x_n^p,$$

where $x_1, x_2, \ldots, x_n$ are the roots of $x^n + c_1x^{n-1} + \cdots + c_n = 0$. Then

$$\begin{array}{ccccccccccc}
 & & & & & & S_1 & + & c_1 & = & 0, \\
 & & & & S_2 & + & c_1S_1 & + & 2c_2 & = & 0, \\
 & & & & & & & & & \vdots & \\
S_n & + & c_1S_{n-1} & + & \cdots & + & c_{n-1}S_1 & + & nc_n & = & 0.
\end{array}$$

and

$$S_p + c_1S_{p-1} + \cdots + c_nS_{p-n} = 0$$

for $p > n$.

Example 17 *If*

$$\begin{array}{ccccccc} x & + & y & + & z & = & 1, \\ x^2 & + & y^2 & + & z^2 & = & 2, \\ x^3 & + & y^3 & + & z^3 & = & 3, \end{array}$$

determine the value of $x^4 + y^4 + z^4$.

Solution Let $P(t) = (t-x)(t-y)(t-z) = t^3 + c_1t^2 + c_2t + c_3$. The relevant formulas are

$$\begin{array}{ccccccccc} & & & & S_1 & + & c_1 & = & 0, \\ & & S_2 & + & c_1S_1 & + & 2c_2 & = & 0, \\ S_3 & + & c_1S_2 & + & c_2S_1 & + & 3c_2 & = & 0, \\ S_4 & + & c_1S_3 & + & c_2S_2 & + & c_3S_1 & = & 0. \end{array}$$

Substituting $S_1 = 1, S_2, = 2, S_3 = 3$ and solving for c_1, c_2, c_3, using the first three equations, we find $c_1 = -1, c_2 = -1/2, c_3 = -1/6$. Thus x, y, z are the roots of $t^3 - t^2 - \frac{1}{2}t - \frac{1}{6} = 0$. We don't have to solve this equation in order to find S_4. Simply note that the fourth equation now reads $S_4 - 3 - 1 - 1/6 = 0$. Thus $x^4 + y^4 + z^4 = 25/6$. ■

Example 18 *Find all solutions of the system of equations*

$$\begin{array}{ccccccc} x & + & y & + & z & = & 0, \\ x^2 & + & y^2 & + & z^2 & = & 6ab \\ x^3 & + & y^3 & + & z^3 & = & 3(a^3+b^3). \end{array}$$

Solution The key is that the left side of each equation is a symmetric function of x, y, z. This suggests that we can use the information given to construct a polynomial equation whose roots are x, y, z. Let

$$P(t) = (t-x)(t-y)(t-z) = t^3 + c_1t^2 + c_2t + c_3.$$

Then $c_1 = -(x+y+z) = 0$ and

$$c_2 = xy + yz + zx = \frac{(x+y+z)^2 - (x^2+y^2+z^2)}{2} = -3ab.$$

Moreover, $P(x) = P(y) = P(z) = 0$ yields $(x^3+y^3+z^3) + c_1(x^2+y^2+z^2) + c_2(x+y+z) + 3c_3 = 0$. We find

$$c_3 = -\frac{x^3+y^3+z^3}{3} = -(a^3+b^3).$$

Thus x, y, z are the roots of the cubic equation $t^3 - 3abt - (a^3+b^3) = 0$. Observe that $t = a+b$ is one of the roots; we can factor to obtain $[t-(a+b)][t^2+(a+b)t+(a^2-ab+b^2)]$, and so find the complete solution set:

$$S = \{a+b,\ a\omega + b\overline{\omega},\ a\overline{\omega} + b\omega\}.$$

(As before, ω and $\overline{\omega}$ are the two complex cubic roots of unity.) ■

By Newton's formulas, all power sums of the roots of the equation $x^n + c_1x^{n-1} + \cdots + c_n = 0$ are real if all of the coefficients $c_1, c_2, \ldots, c_n$ are real.

Example 19 (1980 USAMO) *Let $G_n = a^n \sin(nA) + b^n \sin(nB) + c^n \sin(nC)$, where a, b, c, A, B, C are real numbers and $A + B + C$ is a multiple of π. Prove that if $G_1 = G_2 = 0$, then $G_n = 0$ for every natural number n.*

Solution Let $z_1 = a(\cos A + i \sin A)$, $z_2 = b(\cos B + i \sin B)$, and $z_3 = c(\cos C + i \sin C)$. Using De Moivre's formula, we see that since $G_1 = G_2 = 0$ and $A + B + C$ is a multiple of π,

$$\operatorname{Im}(z_1 + z_2 + z_3) = \operatorname{Im}(z_1^2 + z_2^2 + z_3^2) = \operatorname{Im}(z_1 z_2 z_3) = 0.$$

Thus σ_1 and σ_3 are real. Since $S_1 = z_1 + z_2 + z_3$ and $S_2 = z_1^2 + z_2^2 + z_3^2$ are real, so is

$$\sigma_2 = z_1 z_2 + z_2 z_3 + z_3 z_1 = \frac{S_1^2 - S_2}{2}.$$

Thus $\sigma_1, \sigma_2, \sigma_3$ are real and $P(z) = (z - z_1)(z - z_2)(z - z_3)$ is a polynomial with real coefficients. Hence $S_n = z_1^n + z_2^n + z_3^n$ is real for all $n \geq 0$. This gives us the desired conclusion since $G_n = \operatorname{Im} S_n$ ■.

Exercise 20 *Let a, b, c be real numbers such that*

$$\begin{array}{ccccccc} a & + & b & + & c & = & 3, \\ a^2 & + & b^2 & + & c^2 & = & 5, \\ a^3 & + & b^3 & + & c^3 & = & 7. \end{array}$$

Find $x^4 + y^4 + z^4$.

Exercise 21 *Suppose that all the coefficient of the polynomial $P(x) = c_0 x^n + c_1 x^{n-1} + \cdots + x^2 + x + 1$ are real numbers. Show that $P(x)$ has at least one non-real zero.*

Exercise 22 (1991 MAΘ) *Let r, s, and t be the roots of $x^3 - 6x^2 + 5x - 7 = 0$. Find*

$$\frac{1}{r^2} + \frac{1}{s^2} + \frac{1}{t^2}.$$

Exercise 23 *Let a, b, c be real numbers such that $a + b + c = 0$. Show that*

$$\frac{a^5 + b^5 + c^5}{5} = \left(\frac{a^2 + b^2 + c^2}{2}\right)\left(\frac{a^3 + b^3 + c^3}{3}\right).$$

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