Geometric Inequalities Marathon [1]

The First 100 Problems and Solutions

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1 Preface

On Wednesday, April 20, 2011, at 8:00 PM, I was inspired by the existing Mathlinks marathons to create a marathon on Geometric Inequalities - the fusion of the beautiful worlds of Geometry and Multivariable Inequalities. It was the result of the need for expository material on GI techniques, such as the crucial Rrs, which were well-explored by only a small fraction of the community. Four months later, the thread has over 100 problems with full solutions, and not a single pending problem. On Friday, August 26, 2011, at 5:30 PM, I locked the thread indefinitely with the following post:

The reason is that most of the known techniques have been displayed, which was my goal. Recent problems are tending to to be similar to old ones or they require methods that few are capable of utilizing at this time. Until the community is ready for a new wave of more diffcult GI, and until more of these new generation GI have been distributed to the public (through journals, articles, books, internet, etc.), this topic will remain locked.

This collection is a tribute to our hard work over the last few months, but, more importantly, it is a source of creative problems for future students of GI. My own abilities have increased at least several fold since the exposure to the ideas behind these problems, and all those who strive to find proofs independently will find themselves ready to tackle nearly any geometric inequality on an olympiad or competition.

The following document is dedicated to my friends Constantin Mateescu and Réda Afare (Thalesmaster), and the pioneers Panagiote Ligouras and Virgil Nicula, all four of whom have contributed much to the evolution of GI through the collection and creation of GI on Mathlinks.

To report errors, a Mathlinks PM can be sent BigSams, or an email to `samer_seraj@hotmail.com`.

Samer Seraj
September 4, 2011

[1] The original thread: `http://www.artofproblemsolving.com/Forum/viewtopic.php?f=151&t=403006/`

2 Notation

For a $\triangle ABC$:

- Let $AB = c$, $BC = a$, $CA = b$ be the sides of $\triangle ABC$.
- Let $A = m(\angle BAC)$, $B = m(\angle ABC)$, $C = m(\angle BCA)$ be measures of the angles of $\triangle ABC$.
- Let Δ be the area of $\triangle ABC$.
- Let P be any point inside $\triangle ABC$, and let Q be an arbitrary point in the plane. Let the cevians through P and A, B, C intersect a, b, c at P_a, P_b, P_c respectively.
- Let the distance from P to a, b, c, extended if necessary, be d_a, d_b, d_c respectively.
- Let arbitrary cevians issued from A, B, C be d, e, f respectively.
- Let the semiperimeter, inradius, and circumradius be s, r, R respectively.
- Let the heights issued from A, B, C be h_a, h_b, h_c respectively, which meet at the orthocenter H.
- Let the feet of the perpendiculars from H to BC, CA, AB be H_a, H_b, H_c respectively.
- Let the medians issued from A, B, C be m_a, m_b, m_c respectively, which meet at the centroid G.
- Let the midpoints of A, B, C be M_a, M_b, M_c respectively.
- Let the internal angle bisectors issued from A, B, C be l_a, l_b, l_c respectively, which meet at the incenter I, and intersect their corresponding opposite sides at L_a, L_b, L_c respectively.
- Let the feet of the perpendiculars from I to BC, CA, AB be Γ_a, Γ_b, Γ_c respectively.
- Let the centers of the excircles tangent to BC, CA, AB be I_a, I_b, I_c respectively, and the excircles be tangent to BC, CA, AB at E_a, E_b, E_c.
- Let the radii of the excircles tangent to BC, CA, AB be r_a, r_b, r_c respectively.
- Let the symmedians issued from A, B, C be s_a, s_b, s_c respectively, which meet at the Lemoine Point S, and intersect their corresponding opposite sides at S_a, S_b, S_c respectively.
- Let Γ be the Gergonne Point, and the Gergonne cevians through A, B, C be g_a, g_b, g_c respectively.
- Let N be the Nagel Point, and the Nagel cevians through A, B, C be n_a, n_b, n_c respectively.

Let $[X]$ denote the area of polygon X.

All $\sum$ and $\prod$ symbols without indices are cyclic.

$\square$ denotes the end of a proof, either for a lemma or the original problem.

3 Problems

1. For $\triangle ABC$, prove that $R \geq 2r$. (**Euler's Inequality**)
2. For $\triangle ABC$, prove that $\sum AB > \sum PA$.
3. For $\triangle ABC$, prove that $\dfrac{ab+bc+ca}{4\Delta^2} \geq \sum \dfrac{1}{s(s-a)}$.
4. For $\triangle ABC$, prove that $r(4R+r) \geq \sqrt{3}\Delta$.
5. For $\triangle ABC$, prove that $\cos \dfrac{B-C}{2} \geq \sqrt{\dfrac{2r}{R}}$.
6. For $\triangle ABC$, prove that $\sqrt{12(R^2 - Rr + r^2)} \geq \sum AI \geq 6r$.
7. A circle with center I is inscribed inside quadrilateral $ABCD$. Prove that $\sum AB \geq \sqrt{2} \cdot \sum AI$.
8. For $\triangle ABC$, prove that $9R^2 \geq \sum a^2$. (**Leibniz's Inequality**)
9. Prove that for any non-degenerate quadrilateral with sides a, b, c, d, it is true that $\dfrac{a^2+b^2+c^2}{d^2} \geq \dfrac{1}{3}$.
10. For $\triangle ABC$, prove that $3 \cdot \sum a \sin A \geq \left(\sum a\right) \cdot \left(\sum \sin A\right) \geq 3(a \sin C + b \sin B + c \sin A)$.
11. For acute $\triangle ABC$, prove that $\sum \cot^3 A + 6 \cdot \prod \cot A \geq \sum \cot A$.
12. For $\triangle ABC$, prove that $\left(\sum \cos \dfrac{A}{2}\right) \cdot \left(\sum \csc \dfrac{A}{2}\right) \geq 6\sqrt{3} + \sum \cot \dfrac{A}{2}$.
13. A 2-dimensional plane is partitioned into x regions by three families of lines. All lines in a family are parallel to each other. What is the least number of lines to ensure that $x \geq 2010$. (**Toronto 2010**)
14. For $\triangle ABC$, prove that $3\sqrt{3}R \geq 2s$.
15. For $\triangle ABC$, prove that $\sum \dfrac{1}{2-\cos A} \geq 2 \geq 3 \cdot \sum \dfrac{1}{5-\cos A}$.
16. For $\triangle ABC$, prove that $\dfrac{1}{8} \geq \prod \sin \dfrac{A}{2}$.
17. In right-angled $\triangle ABC$ with $\angle A = 90^\circ$, prove that $\dfrac{3\sqrt{3}}{4} \cdot a \geq h_a + \max\{b, c\}$.
18. For $\triangle ABC$, prove that $s \cdot \sum h_a \geq 9\Delta$ with equality holding if and only if $\triangle ABC$ is equilateral.
19. Prove that the semiperimeter of a triangle is greater than or equal to the perimeter of its orthic triangle.
20. Prove that of all triangles with same base and area, the isosceles triangle has the least perimeter.
21. $ABCD$ is a convex quadrilateral with area 1. Prove that $AC + BD + \sum AB \geq 4 + 2\sqrt{2}$.
22. For $\triangle ABC$, prove that $\sum \csc \dfrac{A}{2} \geq 4\sqrt{\dfrac{R}{r}}$.
23. For $\triangle ABC$, prove that $\sum \sin^2 \dfrac{A}{2} \geq \dfrac{3}{4}$.

24. Of all triangles with a fixed perimeter, dtermine the triangle with the greatest area.

25. Let $ABCD$ be a parallelogram such that $\angle A \leq 90$. Altitudes from A meet BC, CD at E, F respectively. Let r be the inradius of $\triangle CEF$. Prove that $AC \geq 4r$. Determine when equality holds.

26. For $\triangle ABC$, the feet of the altitudes from B, C to AC, AB respectively, are E, D respectively. Let the feet of the altitudes from D, E to BC be G, H respectively. Prove that $DG + EH \leq BC$. Determine when equality holds.

27. For $\triangle ABC$, a line l intersects AB, CA at M, N respectively. K is a point inside $\triangle ABC$ such that it lies on l. Prove that $\Delta \geq 8 \cdot \sqrt{[BMK] + [CNK]}$.

28. For $\triangle ABC$, prove that $\sqrt{\frac{15}{4} + \sum \cos(A - B)} \geq \sum \sin A$.

29. Let p_I be the perimeter of the Intouch/Contact Triangle of $\triangle ABC$. Prove that $p_I \geq 6r\left(\frac{s}{4R}\right)^{\frac{1}{3}}$.

30. In addition to $\triangle ABC$, let $\triangle A'B'C'$ be an arbitrary triangle. Prove that $1 + \frac{R}{r} \geq \sum \frac{\sin A}{\sin A'}$.

31. For $\triangle ABC$, prove that $\sum \cos^2 \frac{B-C}{2} \geq 24 \cdot \prod \sin \frac{A}{2}$.

32. For $\triangle ABC$, prove that $\sum h_a \geq 9r$.

33. For $\triangle ABC$, prove that $\sum \cos \frac{A-B}{2} \geq \sum \sin \frac{3A}{2}$.

34. For $\triangle ABC$, prove that $\sum \sin^2 \frac{A}{2} + \prod \cos \frac{B-C}{2} \geq 1$.

35. For $\triangle ABC$, AO, BO, CO are extended to meet the circumcircles of $\triangle BOC, \triangle COA, \triangle AOB$ respectively, at K, L, N respectively. Prove that $\frac{AK}{OK} + \frac{BL}{OL} + \frac{CM}{OM} \geq \frac{9}{2}$.

36. For $\triangle ABC$, prove that $\frac{9abc}{a+b+c} \geq 4\sqrt{3}\Delta$.

37. For $\triangle ABC$, prove that $\sum a^2 b(a - b) \geq 0$.

38. Show that for all $0 < a, b < \frac{\pi}{2}$ we have $\frac{\sin^3 a}{\sin b} + \frac{\cos^3 a}{\cos b} \geq \sec(a - b)$

39. For all parallelograms with a given perimeter, explicitly define those with the maximum area.

40. Show that the sum of the lengths of the diagonals of a parallelogram is less than or equal to the perimeter of the parallelogram.

41. For $\triangle ABC$, the parallels through P to AB, BC, CA meet BC, CA, AB respectively, at L, M, N respectively. Prove that $\frac{1}{8} \geq \frac{AN}{NB} \cdot \frac{BL}{LC} \cdot \frac{CM}{MA}$.

42. For $\triangle ABC$, prove that $\sum a \sin \frac{A}{2} \geq s$

43. For $\triangle ABC$, it is true that $BC = CA$ and $BC \perp CA$. P is a point on AB, and Q, R are the feet of the perpendiculars from P to BC, CA respectively. Prove that regardless of the location of P, $\max\{[APR], [BPQ], [PQCR]\} \geq \frac{4}{9}\Delta$. (**Generalization of Canada 1969**)

44. For $\triangle ABC$, prove that $\sum a^2 + \dfrac{abc}{\sqrt{3}R} \geq 4(abc)^{\frac{2}{3}}$.

45. For $\triangle ABC$, prove that $6R \geq \sum \dfrac{a^2+b^2}{m_c^2}$.

46. For a convex hexagon $ABCDEF$ with $AB = BC, CD = DE, EF = FA$, prove that $\dfrac{BC}{BE} + \dfrac{DE}{DA} + \dfrac{FA}{FC} \geq \dfrac{3}{2}$. Determine when equality holds.

47. For $\triangle ABC$, prove that $s\sqrt{3} \geq \sum l_a$.

48. For $\triangle ABC$, prove that $R - 2r \geq \dfrac{1}{12} \cdot \left(2 \cdot \sum m_a - \dfrac{\sum ab}{R}\right)$.

49. For $\triangle ABC$, prove that $\sum a^2 \geq 4\sqrt{3}\Delta \cdot \max\left\{\dfrac{m_a}{h_a}, \dfrac{m_b}{h_b}, \dfrac{m_c}{h_c}\right\}$.

50. $A_1A_2B_1B_2C_1C_2$ is a hexagon with $A_1B_2 \cap C_1A_2 = A$, $B_1C_2 \cap A_1B_2 = B$, $C_1A_2 \cap B_1C_2 = C$ and $AA_1 = AA_2 = BC$, $BB_1 = BB_2 = CA$, $CC_1 = CC_2 = AB$. Prove that $[A_1A_2B_1B_2C_1C_2] \geq 13 \cdot [ABC]$.

51. For $\triangle ABC$, let r_1, r_2 denote the inradii of $\triangle ABM_a, \triangle ACM_a$. Prove that $\dfrac{1}{r_1} + \dfrac{1}{r_2} \geq 2 \cdot \left(\dfrac{1}{r} + \dfrac{2}{a}\right)$.

52. For $\triangle ABC$, prove that $\sum \csc^2 \dfrac{A}{2} \geq \sum \cos(A-B) + 9 \geq 8 \cdot \sum \cos A$.

53. For $\triangle ABC$, find the minimum of the expression $\dfrac{2s^4 - \sum a^4}{\Delta^2}$.

54. For $\triangle ABC$, prove that $\dfrac{\sqrt{3}}{2} \cdot \sum \cos \dfrac{B-C}{4} \geq \sum \cos \dfrac{A}{2}$.

55. For $\triangle ABC$, prove that $3 \cdot \sum a^2 > \Delta \cdot \left(\sum \cot \dfrac{A}{2}\right)^2$.

56. For $\triangle ABC$, $c \leq b \leq a$. Through interior point P and the vertices A, B, C, lines are drawn meeting the opposite sides at X, Y, Z respectively. Prove that $AX + BY + CZ < 2a + b$.

57. For $\triangle ABC$, prove that $\dfrac{s^3}{2abc} \geq \sum \cos^4 \dfrac{A}{2}$.

58. For $\triangle ABC$, let $PA = x, PB = y, PC = z$. Prove that $ayz + bzx + cxy \geq abc$, with equality holding if and only if $P \equiv O$. (**China 1998**)

59. For $\triangle ABC$, prove that $3 \cdot \sum d_a^2 \geq \sum PA^2 \sin^2 A$.

60. For $\triangle ABC$, if $CA + AB > 2 \cdot BC$, then prove that $\angle ABC + \angle ACB > \angle BAC$. (Euclid Contest 2010)

61. For $\triangle ABC$, prove that $\dfrac{\sqrt{7 \cdot \sum a^2 + 2 \cdot \sum ab}}{2} \geq \sum m_a$. (**Dorin Andrica**)

62. For $\triangle ABC$, prove that $\sum \cos \dfrac{A}{2} \geq \dfrac{\sqrt{2}}{2} + \sqrt{\dfrac{1}{2} + (3\sqrt{3} - 2\sqrt{2}) \cdot \dfrac{s}{2R}}$.

63. For $\triangle ABC$, prove that $\dfrac{\sqrt{\sum a^2b^2}}{2\Delta} \geq \max\left\{\dfrac{a}{b} + \dfrac{b}{a}, \dfrac{b}{c} + \dfrac{c}{b}, \dfrac{c}{a} + \dfrac{a}{c}\right\}$.

64. For $\triangle ABC$, prove the following and determine which is stronger: (**Samer Seraj**)

   (a) $\Delta \geq r \cdot \sqrt{\frac{1}{3} \cdot \sum m_a m_b + \frac{1}{2} \cdot \sum ab}$.

   (b) $\Delta \geq r \cdot \sqrt{\frac{2}{3} \cdot \sum m_a m_b + r(r+4R)}$.

65. For any convex pentagon $A_1A_2A_3A_4A_5$, prove that $\sum_{i=1}^{5} (A_iA_{i+2} + A_{i+1}A_{i+4}) > \sum_{i=1}^{5} A_iA_{i_2}^2$. $A_{i+5} \equiv A_i$.

66. For $\triangle ABC$, prove that $s^2 \geq \sum l_a^2$.

67. $ABCD$ is a quadrilateral inscribed in a circle with center O. P is the intersection of its diagonals and R is the intersection of the segments joining the midpoints of the opposite sides. Prove that $OP \geq OR$.

68. For $\triangle ABC$, prove that $\frac{5}{4} \cdot \sum bc > \sum m_b m_c$.

69. For $\triangle ABC$, let $M \in [AC]$, $N \in [BC]$ and $L \in [MN]$, where $[XY]$ denotes the line segment XY. Prove that: $\sqrt[3]{\Delta} \geq \sqrt[3]{S_1} + \sqrt[3]{S_2}$, where $S_1 = [AML]$ and $S_2 = [BNL]$.

70. For $\triangle ABC$, prove that $\sum (b+c)PA \geq 8\Delta$.

71. Right $\triangle ABC$ has hypotenuse AB. The arbitrary point P is on segment CA, but different from the vertices A, C. Prove that $\frac{AB - BP}{AP} > \frac{AB - BC}{CA}$.

72. For $\triangle ABC$, prove that $\max\left\{\frac{BP}{AC}, \frac{CP}{AB}\right\} \geq \sqrt{2} - 1$.

73. For $\triangle ABC$, prove that $\sum \frac{a^2}{s-a} \geq 6\sqrt{3}R$.

74. Let P be a point inside a regular n-gon, with side length s, situated at the distances $x_1, x_2, \ldots, x_n$ from the sides, which are extended if necessary. Prove that $\sum_{i=1}^{n} \frac{1}{x_i} > \frac{2\pi}{s}$.

75. A point A is taken inside an acute angle with vertex O. The line OA forms angles α and β with the sides of the angle. Angle ϕ is given such that $\alpha + \beta + \phi < \pi$. On the sides of the former angle, find points M and N such that $\angle MAN = \phi$, and the area of the quadrilateral $OMAN$ is maximal.

76. For $\triangle ABC$, find the smallest constant k such that it always holds that $k \cdot \sum ab > \sum a^2$.

77. For $\triangle ABC$, prove that $\sum abd_ad_b \leq \frac{4}{3}\Delta^2$, and determine when equality holds.

78. For $\triangle ABC$, let AI, BI, CI extended intersect the circumcircle of $\triangle ABC$ again at X, Y, Z respectively. Prove that $\prod IX \geq \prod AI$.

79. Let $\{a, b, c\} \subset \mathbb{R}^+$ such that $\sum \frac{a^2 + b^2 - c^2}{ab} > 2$. Prove that a, b, c are sides of triangle.

80. Let AP be the internal angle bisector of $\angle BAC$ and suppose Q is the point on segment BC such that $BQ = PC$. Prove that $AQ \geq AP$.

81. For $\triangle ABC$, prove that $\Delta^2 \geq r \cdot \prod l_a$.

82. For $\triangle ABC$, prove that $9R^2 \geq \sum a^2 \geq 18Rr$.

83. For $\triangle ABC$, prove that $\sum (PA \cdot PB \cdot c) \geq abc$.

84. For $\triangle ABC$, prove that $8R^3 \geq \prod IE_a$.

85. For $\triangle ABC$, prove that $\left(\sum \sin \frac{A}{2}\right) \cdot \left(\sum \tan \frac{A}{2}\right) \geq \frac{3\sqrt{3}}{2}$.

86. For $\triangle ABC$, prove that $\sum a^3 + 6abc \geq \left(\sum a\right) \cdot \left(\sum ab\right) > \sum a^3 + 5abc$.

87. D and E are points on congruent sides AB and AC, respectively, of isosceles $\triangle ABC$ such that $AD = CE$. Prove that $2EF \geq BC$. Determine when the equality holds.

88. For $\triangle ABC$, prove that $\sum \frac{AH}{a} \geq 3\sqrt{3}$.

89. Let $M, A_1, A_2, \cdots, A_n$ $(n \geq 3)$, be distinct points in the plane such that $A_1A_2 = A_2A_3 = \cdots A_{n-1}A_n = A_nA_1$. Prove that $\sum_{i=1}^{n-1} \frac{1}{MA_i \cdot MA_{i+1}} \geq \frac{1}{MA_1 \cdot MA_n}$.

90. For $\triangle ABC$, determine $\min\left\{\sum QA^2\right\}$.

91. For $\triangle ABC$, prove that $\frac{\sqrt{8 \cdot \sum a^2 + 4\sqrt{3}\Delta}}{3} \geq \sum GA$.

92. For $\triangle ABC$, prove that $a^2 + b^2 + R^2 \geq c^2$, and determine when equality holds.

93. For $\triangle ABC$, prove that $\left(\sum ab\right) \cdot (s^2 + r^2) \geq 4abcs + 36R^2r^2$.

94. For $\triangle ABC$, prove that $\frac{\sum a^2}{\sum ab} \geq 1 + \sqrt{1 - \frac{2r}{R}}$.

95. For $\triangle ABC$, prove that $\frac{\sin B}{\sin^2 \frac{C}{2}} + \frac{\sin C}{\sin^2 \frac{B}{2}} \geq \frac{4\cos\frac{A}{2}}{1 - \sin\frac{A}{2}}$.

96. In $\triangle ABC$, the internal angle bisectors of angles A, B, C intersect the circumcircle of $\triangle ABC$ again at X, Y, Z respectively. Prove that $AX + BY + CZ > a + b + c$. (**Australia 1982**)

97. An arbitrary line ℓ through the incenter I of $\triangle ABC$ cuts $\overline{AB}$ and $\overline{AC}$ at M and N. Show that $\frac{a^2}{4bc} \geq \frac{BM}{AM} \cdot \frac{CN}{AN}$.

98. For $\triangle ABC$, prove that $\sum GA \geq \sqrt{\frac{2 \cdot \sum a^2 + 4\sqrt{3}\Delta}{3}}$. (A sequel to **Problem 91**)

99. For $\triangle ABC$, prove that $3 \geq \sum \frac{SA}{GA}$.

100. Let $m \in \mathbb{R}^+$ and $\phi \in (0, \pi)$. For $\triangle ABC$, prove that

$$(1 - m\cos\phi) \cdot a^2 + m\,(m - \cos\phi) \cdot b^2 + m\cos\phi \cdot c^2 \;\geq 4m\sin\phi \cdot \Delta$$

Equality holds if and only if $m = \frac{a}{b}$ and $\phi = C$.
For $m = 1$ and $\phi = 60^\circ$ obtain Weitzenböck's Inequality. (**Virgil Nicula**)

4 Solutions

1. **Euler's Original Proof**
   $R(R-2r) = OI^2 \geq 0 \iff R \geq 2r$. $\square$

1. **Author: tonypr**

   Rewrite the inequality as $1 + \dfrac{r}{R} \leq \dfrac{3}{2}$. Then note the identity $1 + \dfrac{r}{R} = \cos A + \cos B + \cos C$.
   So it is sufficient to prove that $2\cos A + 2\cos B + 2\cos C \leq 3$.
   It's easy to verify that this inequality is equivalent to $(1 - (\cos B + \cos C))^2 + (\sin B - \sin C)^2 \geq 0$, which is true by the Trivial Inequality. $\square$

1. **Author: BigSams**

   For positive reals x, y, z it is true that $(x+y)(y+z)(z+x) \geq 8xyz$ by AM-GM: $\prod \dfrac{x+y}{2} \geq \prod \sqrt{xy} = xyz$. By Ravi Substitution, let a, b, c be side lengths of a triangle such that $a = x+y, b = y+z, c = z+x$. The inequality becomes $abc \geq 8(s-a)(s-b)(s-c)$. By Heron's Theorem, the inequality is $sabc \geq 8S^2 \iff \dfrac{abc}{4\Delta} \geq \dfrac{2\Delta}{s}$. Using the fact that $\Delta = \dfrac{abc}{4R} = sr$, $R \geq 2r$. $\square$

1. **Author: BigSams**

   Note that $\sum r_a = 4R + r$ and $\sum \dfrac{1}{r_a} = \dfrac{1}{r}$.
   By CS, $\dfrac{4R+r}{r} = \left(\sum r_a\right) \cdot \left(\sum \dfrac{1}{r_a}\right) \geq 9 \iff R \geq 2r$. $\square$

2. **Author: 1=2**

   Lemma. $AB + AC > PB + BC$

   Proof. Let the extension of BP intersect AC at N. Then the triangle inequality gives us

   $$PN + NC > PC$$

   $$AB + AN > BN = BP + PN$$

   Adding NC to both sides of the second inequality gives $AB+AN+NC > BP+PN+NC > PB+PC$. Note that $AN + NC = AC$, since N is on AC. Therefore $AB + AC > PB + PC$. $\square$

   This lemma implies that $\begin{cases} AB + AC > PB + PC \\ BA + BC > PA + PC \\ CA + CB > PA + PB \end{cases}$

   If we add all three inequalities together, we get $2(AB+BC+AC) > 2(PA+PB+PC)$, which implies the desired result. $\square$

3. **Author: Goutham**

   Let $x = s - a$, $y = s - b$, $z = s - c$ all greater than 0, and $s = x + y + z$, $\Delta^2 = xyzs$.
   We have $\sum x^2 \geq \sum xy \implies \sum(x^2 + 3xy) \geq 4\sum xy$.
   But $\sum(x^2 + 3xy) = \sum(x + y)(x + z) = \sum ab$.
   And so, $\dfrac{\sum ab}{4xyzs} \geq \dfrac{\sum xy}{xyzs}$. Therefore, we have $\dfrac{\sum ab}{4\Delta} \geq \sum \dfrac{1}{s(s-a)}$. $\square$

4. **Author: Mateescu Constantin**

   Using the well-known formula for area i.e. $\Delta = sr$, the inequality rewrites as: $s\sqrt{3} \leq 4R + r$ $(*)$. Of course, this is weaker than Gerretsen's Inequality i.e. $s^2 \leq 4R^2 + 4Rr + 3r^2$, since the inequality: $4R^2 + 4Rr + 3r^2 \leq \dfrac{(4R + r)^2}{3}$ reduces to Euler's inequality i.e. $R \geq 2r$. However, there is also a simple method to obtain directly the inequality $(*)$. In the well known inequality:
   $3(xy + yz + zx) \leq (x + y + z)^2$ we take: $\left\Vert \begin{array}{l} x = (s-b)(s-c) \\ \\ y = (s-c)(s-a) \\ \\ z = (s-a)(s-b) \end{array} \right\Vert$ and thus we obtain:
   $3s(s-a)(s-b)(s-c) \leq [r(4R + r)]^2$, whence $\sqrt{3}\Delta \leq r(4R + r) \iff (*)$. $\square$

5. **Author: Thalesmaster**

   Note the identities $\begin{cases} \cos\dfrac{B-C}{2} = \cos\dfrac{B}{2}\cos\dfrac{C}{2} + \sin\dfrac{B}{2}\sin\dfrac{C}{2} \\ \\ \cos\dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{bc}} \\ \\ \sin\dfrac{B}{2} = \sqrt{\dfrac{s(s-a)}{bc}} \end{cases}$ and $\begin{cases} r = \dfrac{\Delta}{s} \\ \\ R = \dfrac{abc}{4\Delta} \end{cases}$

   Using Ravi's substitution: $\begin{cases} a = y + z \\ b = z + x \\ c = x + y \end{cases}$, the inequality is equivalent to: $(2x + y + z)^2 \geq 8x(y + z)$, which is true according to AM-GM Inequality. $\square$

6. **Author: FantasyLover**

   Right Side.
   Let (I) meet sides AB, BC, CA at P, Q, R, respectively. Furthermore, denote by a, b, c the lengths of AR, BP, CQ.
   The given inequality is equivalent to $\sqrt{a^2 + r^2} + \sqrt{b^2 + r^2} + \sqrt{c^2 + r^2} \geq 6r$. On the other hand, $r(a + b + c) = \sqrt{abc(a + b + c)} \iff r = \sqrt{\dfrac{abc}{a + b + c}}$ from Heron's Formula.
   Hence, it suffices to prove $\sum\sqrt{a^2 + \dfrac{abc}{a + b + c}} \geq 6\sqrt{\dfrac{abc}{a + b + c}} \iff \sum\sqrt{a(a + b)(a + c)} \geq 6\sqrt{abc}$.
   However, using AM-GM Inequality twice gives $\sum\sqrt{a(a + b)(a + c)} \geq 3\sqrt[6]{abc(a + b)^2(b + c)^2(c + a)^2} \geq$

$3\sqrt[6]{abc\cdot 64(abc)^2} \geq 6\sqrt{abc}$, as desired. $\square$

Left Side.
Lemma. $AI + BI + CI \leq 2(R+r)$ (**Author: Mateescu Constantin**)

Proof. Show easily that $AI = \dfrac{bc}{s}\cdot\cos\dfrac{A}{2} = \dfrac{1}{s}\cdot\sqrt{bc}\cdot\sqrt{s(s-a)}$ a.s.o. Thus, we have: $\left(\sum AI\right)^2 = \dfrac{1}{s^2}\cdot\left(\sum\sqrt{bc}\cdot\sqrt{s(s-a)}\right)^2 \overset{C.B.S.}{\leq} \dfrac{1}{s^2}\cdot(ab+bc+ca)\cdot\sum s(s-a) = ab+bc+ca \ \leq\ 4(R+r)^2$.
The last inequality is due to Gerretsen i.e. $s^2 \ \leq\ 4R^2+4Rr+3r^2$. Therefore, we have shown that: $AI+BI+CI \ \leq\ 2(R+r)$. $\square$

As a direct consequence of the lemma, it suffices to prove $2(R+r) \ \leq\ 2\sqrt{3(R^2-Rr+r^2)} \iff 2R^2-5Rr+2r^2 \geq 0$.
However, ths is equivalent to $(2R-r)(R-2r)\geq 0$, which is indeed true. $\square$
For both inequalities, equality holds for $\triangle ABC$ equilateral.

6. **Author: Thalesmaster**

Left Side.
Note that: $\begin{cases} AI^2 = bc-4Rr \\ BI^2 = ca-4Rr \\ CI^2 = ab-4Rr \end{cases}$ According to C.S Inequality:
$3(AI^2+BI^2+CI^2) \geq (AI+BI+CI)^2 \iff \sqrt{3(s^2+r^2-8Rr)} \geq AI+BI+CI$
So it suffices to show that
$\sqrt{3(s^2+r^2-8Rr)} \leq \sqrt{12(R^2-Rr+r^2)} \Leftrightarrow s^2+r^2+8Rr \leq 4R^2-4Rr+4r^2$
$\Leftrightarrow s^2 \leq 4R^2+4Rr+3r^2$, which is the Gerretsen Inequality. $\square$

6. **Author: tonypr**

Right Side.
Note that $AI = \dfrac{r}{\sin\frac{A}{2}}$. Applying this cyclically to BI and CI, the left hand side is equivalent to

$$6r \leq \frac{r}{\sin\frac{A}{2}} + \frac{r}{\sin\frac{B}{2}} + \frac{r}{\sin\frac{C}{2}} \iff 2 \leq \frac{\dfrac{1}{\sin\frac{A}{2}} + \dfrac{1}{\sin\frac{B}{2}} + \dfrac{1}{\sin\frac{C}{2}}}{3}$$

$$2 \leq \frac{\csc\frac{A}{2}+\csc\frac{B}{2}+\csc\frac{C}{2}}{3} \iff \csc\left(\frac{A+B+C}{6}\right) \leq \frac{\csc\frac{A}{2}+\csc\frac{B}{2}+\csc\frac{C}{2}}{3}$$

which follows from Jensen's Inequality since $\csc\dfrac{x}{2}$ is convex for $x\in(0,\pi)$. $\square$

7. **Author: BigSams**

It is well-known that $\angle AID + \angle BIC = 180^\circ$. There are two implications: $\sin\angle AID = \sin\angle BIC$ and $\cos\angle AID = -\cos\angle BIC$. Let r be the inradius.

$$[AID] = \frac{\sin\angle AID\cdot AI\cdot DI}{2} = \frac{AD\cdot r}{2} \implies \frac{AI\cdot DI}{AD} = \frac{r}{\sin\angle AID}.$$

Similarly, $\dfrac{BI\cdot CI}{BC} = \dfrac{r}{\sin\angle BIC}$.

Combining, $\dfrac{AI \cdot DI}{AD} = \dfrac{r}{\sin \angle AID} = \dfrac{r}{\sin \angle BIC} = \dfrac{BI \cdot CI}{BC}$
$\implies \dfrac{AI \cdot DI}{BI \cdot CI} = \dfrac{AD}{BC}$.

By the Cosine Law, $2\cos \angle AID = \dfrac{AI^2 + DI^2 - AD^2}{AI \cdot DI}$ and $2\cos \angle BIC = \dfrac{BI^2 + CI^2 - BC^2}{BI \cdot CI}$.
Combining, $\dfrac{AI^2 + DI^2 - AD^2}{AI \cdot DI} = 2\cos \angle AID = -2\cos \angle BIC = -\dfrac{BI^2 + CI^2 - BC^2}{BI \cdot CI}$
$\implies \dfrac{AI^2 + DI^2 - AD^2}{AI \cdot DI} = \dfrac{BC^2 - BI^2 - CI^2}{BI \cdot CI}$
$\implies \dfrac{AI^2}{AI \cdot DI} + \dfrac{DI^2}{AI \cdot DI} + \dfrac{BI^2}{BI \cdot CI} + \dfrac{CI^2}{BI \cdot CI} = \dfrac{AD^2}{AI \cdot DI} + \dfrac{BC^2}{BI \cdot CI}$

It is well-known that for a tangential quadrilateral, the sum of two opposite sides is equal to the semiperimeter.
So $AB + BC + CD + DA = 2(AD + BC) = \sqrt{2(AD + BC)^2}$
$= \sqrt{4(AD^2 + AD \cdot BC + BC \cdot AD + BC^2)}$
$= \sqrt{4\left(AD^2 + \dfrac{AD^2 \cdot BI \cdot CI}{AI \cdot DI} + \dfrac{BC^2 \cdot AI \cdot DI}{BI \cdot CI} + BC^2\right)}$
$= \sqrt{4\left(\dfrac{AD^2}{AI \cdot DI} + \dfrac{BC^2}{BI \cdot CI}\right) \cdot (AI \cdot DI + BI \cdot CI)}$

By Cauchy-Schwarz, $\dfrac{AI^2}{AI \cdot DI} + \dfrac{DI^2}{AI \cdot DI} + \dfrac{BI^2}{BI \cdot CI} + \dfrac{CI^2}{BI \cdot CI}$
$\geq \dfrac{(AI + BI + CI + DI)^2}{2(AI \cdot DI + BI \cdot CI)}$
$\iff \sqrt{4\left(\dfrac{AD^2}{AI \cdot DI} + \dfrac{BC^2}{BI \cdot CI}\right) \cdot (AI \cdot DI + BI \cdot CI)} \geq \sqrt{2}(AI + BI + CI + DI)$
$\iff AB + BC + CD + DA \geq \sqrt{2}(AI + BI + CI + DI)$ $\square$

8. **Author: RSM**

$$R^2 - \frac{a^2 + b^2 + c^2}{9} = OG^2 \geq 0 \iff 9R^2 \geq \sum a^2. \ \square$$

9. **Author: RSM**

By CS, $\dfrac{a^2 + b^2 + c^2}{3} \geq \left(\dfrac{a + b + c}{3}\right)^2$. By Triangle Inequality, $\left(\dfrac{a + b + c}{3}\right)^2 \geq \dfrac{d^2}{9}$. $\square$

10. **Author: Thalesmaster**

The desired inequality is equivalent to $3(a^2 + b^2 + c^2) \geq (a + b + c)^2 \geq 3(ab + bc + ca)$
$\iff a^2 + b^2 + c^2 \geq ab + bc + ca \iff (a - b)^2 + (b - c)^2 + (c - a)^2 \geq 0$
Which is true, with equality if and only if $a = b = c$. $\square$

11. **Author: Thalesmaster**

Let $\begin{cases} p = \sum \cot A \\ \\ q = \sum \cot B \cot C = 1 \\ \\ r = \prod \cot A \end{cases}$

The inequality is equivalent to:
$(p^3 - 3pq + 3r) + 6r \ge p \iff p^3 - 3pq + 9r \ge pq \iff p^3 - 4pq + 9r \ge 0$
Which is Schur's Inequality. $\square$

12. **Author: gaussintraining**

Using the identities $\begin{cases} \cos\dfrac{A}{2} = \sqrt{\dfrac{s(s-a)}{bc}} \\ \\ \sin\dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{bc}} \\ \\ \sum \cot\dfrac{A}{2} = \dfrac{s}{r} \end{cases}$,

the inequality is equivalent to $\left(\sum\sqrt{\dfrac{s(s-a)}{bc}}\right)\cdot\left(\sum\sqrt{\dfrac{bc}{(s-b)(s-c)}}\right) \ge 6\sqrt{3} + \dfrac{s}{r}$

By Cauchy-Schwarz,

LHS $= \left(\sum\sqrt{\dfrac{s(s-a)}{bc}}\right)\cdot\left(\sum\sqrt{\dfrac{bc}{(s-a)(s-b)}}\right) \ge \left(\sum\sqrt[4]{\dfrac{s(s-a)}{(s-b)(s-c)}}\right)^2 = \left(\sum\dfrac{\sqrt{s-a}}{\sqrt{r}}\right)^2$

using Heron's Formula. Thus, we have to prove

$(\sqrt{s-a} + \sqrt{s-b} + \sqrt{s-c})^2 \ge 6\sqrt{3} + \dfrac{s}{r}$

$\implies 2(\sum\sqrt{s-a}\sqrt{s-b}) \ge 6r\sqrt{3}$

By AM-GM, $\dfrac{\sqrt{(s-a)(s-b)} + \sqrt{(s-b)(s-c)} + \sqrt{(s-c)(s-a)}}{3} \ge \sqrt[3]{(s-a)(s-b)(s-c)}$.

Using Heron's Formula again, we find that $\sqrt[3]{(s-a)(s-b)(s-c)} = \sqrt[3]{r^2 s}$.

Therefore, we finally have to show that $3\sqrt[3]{r^2 s} \ge 3r\sqrt{3} \implies s \ge 3r\sqrt{3}$, which is well-known. $\square$

12. **Author: Thalesmaster**

After applying CS, it suffices to show that $\sum\sqrt{\cot\dfrac{B}{2}\cot\dfrac{C}{2}} \ge 3\sqrt{3}$

Which is true according to AM-GM and Mitrinovic's Inequality:

$\sum\sqrt{\cot\dfrac{B}{2}\cot\dfrac{C}{2}} \ge 3\sqrt[3]{\prod\cot\dfrac{A}{2}} = 3\sqrt[3]{\dfrac{s}{r}} \ge 3\sqrt{3}$. $\square$

13. **Author: applepi2000**

Let s_k denote the number of lines in family k. First, we draw the a and b families. It is not hard to see that there are a maximum of $(s_a+1)(s_b+1)$ regions. Now when we add each line from family c, it intersects a maximum of $s_a + s_b$ times, creating $s_a + s_b + 1$ new regions. Thus, the total number of regions is $s_c(s_a + s_b + 1) + (s_a+1)(s_b+1) = \sum s_a s_b + \sum s_a + 1$.

Let $s_a + s_b + s_c = n$. Then the number of lines is $2010 \le \frac{n^2}{3} + n + 1$. Thus, $n \ge 77$. Indeed, plugging in $s_a = s_b = 26, s_c = 25$ works, so our answer is 77. $\square$

14. **Author: mcrasher**

Since $\sum \sin A = \frac{s}{R}$, it suffices to show that $\sum \sin A \le \frac{3\sqrt{3}}{2}$, which is true by Jensen's Inequality. $\square$

15. **Author: BigSams**

Left Side.
By Euler's Inequality, $2r \le R \iff 8r^2 \le 4Rr \iff 4R^2 + 4Rr + 3r^2 \le 8Rr - 5r^2 + 4R^2$.
By Gerretsen's Inequality, $s^2 \le 4R^2 + 4Rr + 3r^2$.
Combining, $\iff s^2 \le 8Rr - 5r^2 + 4R^2$.
$\iff 4\left(1 + \frac{r}{R}\right) + 2\left(\frac{s^2 - (2R+r)^2}{4R^2}\right) \ge 4 + 3\left(\frac{s^2 + r^2 - 4R^2}{4R^2}\right)$
$\iff 4\sum \cos A + 2\prod \cos A \ge 4 + 3\sum \cos A \cdot \cos B$
$\iff \sum (2 - \cos A) \cdot (2 - \cos B) \ge 2\prod (2 - \cos A) \iff \sum \frac{1}{2 - \cos A} \ge 2$ $\square$

Right Side.
By Euler's Inequality, $2r \le R \iff \frac{72Rr - 9r^2}{5} \le 16Rr - 5r^2$.
By Gerretsen's Inequality, $16Rr - 5r^2 \le s^2$. Combining, $\iff \frac{72Rr - 9r^2}{5} \le s^2$
$\iff 20\left(1 + \frac{r}{R}\right) + 2\left(\frac{s^2 - (2R+r)^2}{4R^2}\right) \le 25 + 7\left(\frac{s^2 + r^2 - 4R^2}{4R^2}\right)$
$\iff 20\sum \cos A + 2\prod \cos A \le 25 + 7\sum \cos A \cdot \cos B$
$\iff \frac{\sum (5 - \cos A) \cdot (5 - \cos B)}{\prod (5 - \cos A)} \le \frac{2}{3} \iff \sum \frac{1}{5 - \cos A} \le \frac{2}{3}$. $\square$

16. **Author: Mateescu Constantin**

Using the relation: $\prod \sin \frac{A}{2} = \frac{r}{4R}$, the inequality reduces to $2r \le R$, which is due to Euler. $\square$

17. **Author: ftong**

Let $\theta = \angle C$, and assume without loss of generality that $0^\circ \le \theta \le 45^\circ$, or equivalently, $b \ge c$.
Now $h_A = b\sin\theta$, and $a = \frac{b}{\cos\theta}$, so we wish to prove that $\cos\theta(\sin\theta + 1) \le \frac{3\sqrt{3}}{4}$
It seems now that we must use resort calculus to find the maximum of $f(\theta) = \cos\theta(\sin\theta + 1)$ over the given interval.
Taking the derivative, we have $f'(\theta) = 1 - \sin\theta - 2\sin^2\theta$, so that f takes extremal values at $\sin\theta = \frac{1}{2}$ and $\sin\theta = -1$.
We discard the latter because $\sin\theta$ is positive in our interval, so the maximum occurs at $\theta = \frac{\pi}{6}$, at which point $f(\theta) = \frac{3\sqrt{3}}{4}$ as desired. $\square$

18. **Author: BigSams**

By CS, $9 \leq (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$
$= \left(\frac{a+b+c}{2[ABC]}\right)\left(\frac{2[ABC]}{a}+\frac{2[ABC]}{b}+\frac{2[ABC]}{c}\right)$
$= \frac{sh}{[ABC]} \iff 9[ABC] \leq sh$
Equality holds if and only if $a = b = c$, which is derived from the CS equality condition. $\square$

19. **Author: Goutham**

Lemma. In $\triangle ABC, M, N, P$ are points on sides BC, CA, AB respectively such that perimeter of the $\triangle MNP$ is minimal. Then $\triangle MNP$ is the orthic triangle of $\triangle ABC$. (**Author: Farenhajt**)

Proof.
Let M be an arbitrary point on BC, and M' and M'' its reflections about AB and AC respectively. Then, for a given M, the points N, P which minimize the perimeter of $\triangle MNP$ are the intersections of $M'M''$ with AB and AC.
Triangles AMM' and AMM'' are isosceles, hence $\angle M'AM'' = 2\angle A = \text{const}$, thus $M'M''$, i.e. the required perimeter, is minimal when $AM' = AM'' = AM$ is minimal, which is obviously attained if M is the foot of the perpendicular from A to BC $\quad(*)$.
Now we note that the orthic triangle has the property that, when one of its vertices is reflected about the remaining two sides of the initial triangle, the two reflections are collinear with the two remaining vertices of the orthic triangle - which is easy to prove: $\angle MPN = \pi - 2\angle C \wedge \angle MPB = \angle C$.
Therefore the triangle obtained by the argument $(*)$ is indeed the orthic triangle, as claimed. $\square$

Using the lemma, the orthic triangle does not have a greater perimeter than the medial triangle, which has a perimeter equal to the semiperimeter of the original triangle. $\square$

20. **Author: BigSams**

Let $\triangle ABC$ be an arbitrary triangle with a constant area Δ and constant base a. Since the area and a base are constant, then the height h_a with foot on a is also constant since it can be expressed in terms of constants: $\frac{a \cdot h_a}{2} = \Delta \implies h_a = \frac{2X}{a}$.
Let $AB = c, CA = b$. Let h_a intersect $BC = a$ (extended if necessary) at P. Let $PC = a_1, PB = a_2$.
Note that the perimeter is minimized when $b + c$ is minimized, since a is a constant.
Case 1. $\angle B, \angle C \leq 90°$
Note that $a_1 + a_2 = a$. Also by the Pythagorean Theorem, $b = \sqrt{a_1^2 + h_a^2}, c = \sqrt{a_2^2 + h_a^2}$.
By Minkowski's Inequality, $b+c = \sqrt{a_1^2 + h_a^2} + \sqrt{a_2^2 + h_a^2} \geq \sqrt{(a_1 + a_2)^2 + (2h_a)^2} = \sqrt{a^2 + 4h_a^2}$, which is a constant.
Equality holds if and only if $a_1 = a_2 \implies \sqrt{a_1^2 + h_a^2} = \sqrt{a_2^2 + h_a^2} \implies b = c$.
Case 2. One of $\angle B, \angle C > 90°$
In an obtuse $\triangle ABC$, as P moves farther away from B, C, a_1, a_2 both increase, meaning $\sqrt{a_1^2 + h_a^2}, \sqrt{a_2^2 + h_a^2}$ both increase, implying that b, c both grow without bound, so each of these triangles hav Thus, the perimeter for a triangle with a constant area and a constant base is the one where the two variable sides are equal, resulting in an isosceles triangle. $\square$

21. **Author: r1234**

Let O be the point of intersection of the two diagonals. Now $[ABCD] = \frac{1}{2} \cdot AC \cdot BD \cdot \sin \angle ACD$. So $[ABCD] \leq AC \cdot BD$.
Now again $[ABCD] = \frac{1}{2} \cdot AB \cdot BC \cdot \sin B \leq \frac{1}{2} \cdot AB \cdot BC$ similarly we get $[ABCD] \leq \frac{1}{2} \cdot CD \cdot DA$ on the other hand we get other two inequalities $[ABCD] \leq \frac{1}{2} \cdot AB \cdot CD$ and $[ABCD] \leq \frac{1}{2} \cdot BC \cdot AD$.
Adding the last four inequalities we get$(AB + CD)(BC + DA) \geq 4$. This implies that $(AB + BC + CD + DA)^2 \geq 4(AB + CD)(BC + AD) \geq 16$ or $AB + BC + CD + DA \geq 4$.
On the other hand we get $AC \cdot BD \geq 2$ or $(AC + BD)^2 \geq 8$ or $AC + BD \geq 2\sqrt{2}$.
Adding we get $AB + BC + CD + DA + AC + BD \geq 4 + 2\sqrt{2}$. $\square$

22. **Author: Thalesmaster**

Using Ravi's substitution $\begin{cases} a = x + y \\ b = y + z \\ c = z + x \end{cases}$
We have $\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}} = \sqrt{\frac{yz}{(x+y)(x+z)}}$.
So the inequality is equivalent to $\sum \left(\sin \frac{B}{2} \cdot \sin \frac{C}{2} \right) \geq 2 \cdot \sqrt{\prod \sin \frac{A}{2}} \iff \sum \sqrt{\frac{x}{y+z}} \geq 2$
According to Holder's Inequality, $\left(\sum \frac{x}{\sqrt{x(y+z)}} \right)^2 \cdot \left(\sum x^2(y+z) \right) \geq \left(\sum x \right)^3$
$\iff \left(\sum \frac{x}{\sqrt{x(y+z)}} \right)^2 \geq \frac{(x+y+z)^3}{(x+y+z)(xy+yz+zx) - 3xyz}$
It suffices to show that $\frac{(x+y+z)^3}{(x+y+z)(xy+yz+zx) - 3xyz} \geq 4$
$\iff (x+y+z)^3 - 4(x+y+z)(xy+yz+zx) + 9xyz + 3xyz \geq 0$, which is Schur's Inequality. $\square$

23. **Author: professordad**

Using the half angle identites, $\sum \sin^2 \frac{A}{2} = \sum \frac{1 - \cos A}{2} = \frac{3 - \sum \cos A}{2} \geq \frac{3}{4}$. This is equivalent to $\sum \cos A \geq \frac{3}{2}$, which was proven by **tonypr** in his solution to **Problem 1**. $\square$

24. **Author: ryanstone**

The area is $\sqrt{s(s-a)(s-b)(s-c)}$ by Heron's Theorem.
By AM-GM, $\frac{(s-a)+(s-b)+(s-c)}{3} \geq \sqrt[3]{(s-a)(s-b)(s-c)}^{\frac{1}{3}} \iff (s-a)(s-b)(s-c) \geq \frac{s^3}{27}$.
So the maximum value of the area is $\sqrt{\frac{s^4}{27}} = \frac{s^2}{3\sqrt{3}}$, which occurs when $a = b = c$. $\square$

25. **Author: math_explorer**

Since $\angle AEC$ and $\angle AFC$ are both right, the points $AECF$ are cyclic and AC is a diameter. Therefore AC is twice the circumradius of $\triangle CEF$.
By Euler's inequality of a triangle in $\triangle CEF$ the circumradius is at least twice the inradius, so $AC \geq 4r_1$, with equality iff $\triangle CEF$ is equilateral iff $\angle C = 60^\circ$ and A lies on the angle bisector of $\angle ECF$ iff $ABCD$ is a rhombus and $\angle C = 60^\circ$. $\square$

26. **Author: truongtansang89**

Note that $DG \cdot BC = DB \cdot DC \Rightarrow DG \cdot BC = BC^2 \cdot \cos B \sin B \Rightarrow DG = \dfrac{1}{2} BC \sin 2B$.
Similarly, $EH = \dfrac{1}{2} BC \sin 2C \Rightarrow DG + EH = BC \cdot \sin A \cdot \cos(B - C) \leq BC$.
Hence, equality holds when $A = \dfrac{\pi}{2}$ and $B = C = \dfrac{\pi}{4}$. $\square$

27. **Author: Mateescu Constantin**

Let us denote: $\dfrac{AM}{MB} = q$, $\dfrac{AN}{NC} = r$, $\dfrac{MK}{KN} = t$, where $q, r, t > 0$.
Observe that: $\dfrac{[AMN]}{[ABC]} = \dfrac{AM \cdot AN}{bc} = \dfrac{qr}{(q+1)(r+1)}$,
From where: $[AMN] = \dfrac{qr}{(q+1)(r+1)} \cdot [ABC] (*)$. Moreover, we can write the following relations:

$$\begin{cases} \left\| \begin{array}{lcl} \dfrac{[BMK]}{[AMK]} = \dfrac{1}{q} & \Longrightarrow & [BMK] = \dfrac{[AMK]}{q} \\ \\ \dfrac{[AMK]}{[ANK]} = t & \Longrightarrow & [AMK] = \dfrac{t \cdot [AMN]}{t+1} \end{array} \right\| \Longrightarrow [BMK] = \dfrac{t \cdot [AMN]}{q(t+1)} \overset{(*)}{=} \dfrac{rt \cdot [ABC]}{(q+1)(r+1)(t+1)} \\ \\ \left\| \begin{array}{lcl} \dfrac{[CNK]}{[ANK]} = \dfrac{1}{r} & \Longrightarrow & [CNK] = \dfrac{[ANK]}{r} \\ \\ \dfrac{[ANK]}{[AMK]} = \dfrac{1}{t} & \Longrightarrow & [ANK] = \dfrac{[AMN]}{t+1} \end{array} \right\| \Longrightarrow [CNK] = \dfrac{[AMN]}{r(t+1)} \overset{(*)}{=} \dfrac{q \cdot [ABC]}{(q+1)(r+1)(t+1)} \end{cases}$$

Thus, the proposed inequality reduces to: $[ABC] \geq 8 \cdot \sqrt{\dfrac{qrt}{(q+1)^2(r+1)^2(t+1)^2} \cdot [ABC]^2} \iff$ $(q+1)(r+1)(t+1) \geq 8\sqrt{qrt}$, which is clearly true by AM-GM inequality. Equality occurs if and only if $q = r = t = 1$, i.e. $\dfrac{AM}{MB} = \dfrac{AN}{NC} = \dfrac{MK}{KN} = 1$. $\square$

28. **Author: BigSams**

By Euler's Inequality, $R \geq 2r \iff \dfrac{11R^2 + 4Rr + 2r^2}{2} \geq 4R^2 + 4Rr + 3r^2$
By Gerretsen's Inequality, $4R^2 + 4Rr + 3r^2 \geq s^2$.
Combining, $\iff \dfrac{11R^2 + 4Rr + 2r^2}{2} \geq s^2 \iff \dfrac{9}{2} + \left(1 + \dfrac{r}{R}\right)^2 \geq \left(\dfrac{s}{R}\right)^2$.
Using the well-known identities $\begin{cases} \sum \sin A = \dfrac{s}{R} \\ \\ \sum \cos A = 1 + \dfrac{r}{R} \end{cases}$, the above inequality becomes

$\iff \dfrac{9}{2} + \left(\sum \cos A\right)^2 \geq \left(\sum \sin A\right)^2$

$\iff \sum \sin A \leq \sqrt{\dfrac{9}{4} + \dfrac{\left(\sum \cos A\right)^2 + \left(\sum \sin A\right)^2}{2}}$

Note that $\sin^2 A + \cos^2 A = 1 \implies \sum \sin^2 A + \sum \cos^2 A = 3$.

Note that $\cos(A - B) = \cos A \cos B + \sin A \sin B$

$\implies 2 \sum \cos(A - B) = 2 \sum (\cos A \cos B) + 2 \sum (\sin A \sin B)$.

Adding these gives $3 + 2 \sum \cos(A - B)$

$= \sum \sin^2 A + \sum \cos^2 A + 2 \sum (\cos A \cos B) + 2 \sum (\sin A \sin B)$

$= \left(\sum \cos A\right)^2 + \left(\sum \sin A\right)^2$.

So $3 + 2 \sum \cos(A - B) = \left(\sum \cos A\right)^2 + \left(\sum \sin A\right)^2$.

Applying the above identity, the previously derived $\sum \sin A \leq \sqrt{\dfrac{9}{4} + \dfrac{\left(\sum \cos A\right)^2 + \left(\sum \sin A\right)^2}{2}}$ becomes $\iff \sum \sin A \leq \sqrt{\dfrac{15}{4} + \sum \cos(A - B)}$, as desired. $\square$

29. **Author: BigSams**

Let the sides of $\triangle ABC$ be $AB = c, BC = a, CA = b$, with corresponding sides of the intouch circle being a', b', c' respectively.

Note that $\begin{cases} a' = 2(s-a) \sin \dfrac{A}{2} \\ \\ b' = 2(s-b) \sin \dfrac{B}{2} \\ \\ c' = 2(s-c) \sin \dfrac{C}{2} \end{cases}$, and $\begin{cases} \prod (s-a) = sr^2 \\ \\ \prod \sin \dfrac{A}{2} = \dfrac{r}{4R} \end{cases}$

By AM-GM, $s = \sum a' \geq 3 \cdot \left(\prod a'\right)^{\frac{1}{3}} = 3 \cdot \left(\prod 2(s-a) \sin \dfrac{A}{2}\right)^{\frac{1}{3}} = 6r \left(\dfrac{s}{4R}\right)^{\frac{1}{3}}$. $\square$

30. **Author: Thalesmaster**

Let x, y, z be positive real numbers.

Klamkin's Inequality states that $x \sin A' + y \sin B' + z \sin C' \leq \dfrac{1}{2}(xy + yz + zx)\sqrt{\dfrac{x+y+z}{xyz}}$.

For $x = \dfrac{1}{\sin A}$, $y = \dfrac{1}{\sin B}$, $z = \dfrac{1}{\sin C}$, we obtain $\sum \dfrac{\sin A'}{\sin A} \leq \dfrac{1}{2} \dfrac{\sum \sin A}{\prod \sin A} \sqrt{\sum \sin B \sin C}$

$\iff \sum \dfrac{\sin A'}{\sin A} \leq \dfrac{1}{2r}\sqrt{ab + bc + ca}$.

Gerretsen's Inequality gives us $s^2 \leq 4R^2 + 4Rr + 3r^2 \iff ab + bc + ca \leq 4(R + r)^2$

So $\sum \dfrac{\sin A'}{\sin A} \leq \dfrac{2(R + r)}{2r} = 1 + \dfrac{R}{r}$. $\square$

31. **Author: BigSams**

By Euler's Inequality, $R \geq 2r \iff (2R+r)(R-2r) \geq 0 \iff 16Rr - 5r^2 \geq 22Rr - 4R^2 - r^2$.
By Gerretsen's Inequality, $s^2 \geq 16Rr - 5r^2$.

Combining, $s^2 \geq 22Rr - 4R^2 - r^2 \iff \dfrac{3 + \left(1+\frac{r}{R}\right)^2 + \left(\frac{s}{R}\right)^2}{4} \geq 24\left(\dfrac{r}{4R}\right)$

Note the identities: $\begin{cases} \sum \sin A = \dfrac{s}{R} \\ \sum \cos A = 1 + \dfrac{r}{R} \\ \prod \sin \dfrac{A}{2} = \dfrac{r}{4R} \end{cases} \iff 24 \cdot \prod \sin \dfrac{A}{2} \leq \dfrac{3 + (\sum \cos A)^2 + (\sum \sin A)^2}{4}$

$= \dfrac{1}{4} \cdot \left(3 + \sum \cos^2 A + 2\sum \cos A \cdot \cos B + \sum \sin^2 A + 2 \sum \sin A \cdot \sin B\right)$

Note the identities: $\begin{cases} \sin^2 A + \cos^2 A = 1 \\ \cos(A-B) = \cos A \cos B + \sin A \sin B \\ \cos^2 \dfrac{x}{2} = \dfrac{1+\cos x}{2} \end{cases}$

$\iff 24 \cdot \prod \sin \dfrac{A}{2} \leq \sum \dfrac{1 + \cos A \cdot \cos B + \sin A \cdot \sin B}{2} = \sum \dfrac{1+\cos(A-B)}{2} = \sum \cos^2 \dfrac{A-B}{2}$

Thus, $\sum \cos^2 \dfrac{A-B}{2} \geq 24 \cdot \prod \sin \dfrac{A}{2}$. $\square$

32. **Author: applepi2000**

Note that $\Delta = rs$. Let h_i be the altitude to side i. We wish to prove $h_a + h_b + h_c \geq 9r \iff 2\Delta\left(\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}\right) \geq \dfrac{18\Delta}{a+b+c} \iff \dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c} \geq \dfrac{9}{a+b+c}$

Take the reciprocal of both sides, then multiply by 3: $\dfrac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}} \leq \dfrac{a+b+c}{3}$. This is just AM-HM, so we are done. $\square$

33. **Author: Thalesmaster**

After expanding it, the inequality is equivalent to:

$$4 \cdot \left(\sum \sin \frac{A}{2}\right)^3 + \sum \sin \frac{B}{2} \sin \frac{C}{2} + \sum \cos \frac{B}{2} \cos \frac{C}{2} + 12 \cdot \prod \sin \frac{A}{2}$$
$$\geq 12 \cdot \left(\sum \sin \frac{A}{2}\right) \cdot \left(\sum \sin \frac{B}{2} \sin \frac{C}{2}\right) + 3 \cdot \sum \sin \frac{A}{2}$$

Use the substitution: $\begin{cases} X = \dfrac{\pi - A}{2} \\ Y = \dfrac{\pi - B}{2} \\ Z = \dfrac{\pi - C}{2} \end{cases}$, and the identities: $\begin{cases} \sum \cos X = 1 + \dfrac{r}{R} \\ \sum \cos Y \cos Z = \dfrac{s^2 + r^2 - 4R^2}{4R^2} \\ \prod \cos X = \dfrac{s^2 - (2R+r)^2}{4R^2} \\ \sum \sin Y \sin Z = \dfrac{s^2 + r^2 + 4Rr}{4R^2} \end{cases}$

where s, R, r respectively denote the semiperimeter, circumradius and inradius of $\triangle XYZ$.

We find that the previous inequality is equivalent to:

$$4\cdot\left(\sum\cos X\right)^3+\sum\cos Y\cos Z+\sum\sin Y\sin Z+12\cdot\prod\cos X$$
$$\geq 12\cdot\left(\sum\cos X\right)\cdot\left(\sum\cos Y\cdot\cos Z\right)+3\cdot\sum\cos X$$
$$\iff s^2(R-6r)+20R^2r+13Rr^2+2r^3\geq 0$$

If $R\geq 6r$, this is it. If $R\leq 6r$, then it's equivalent to $\dfrac{20R^2r+13Rr^2+2r^3}{6r-R}\geq s^2$ Using the inequality $4R+r\geq\sqrt{3}s$, it suffices to show that: $\dfrac{20R^2r+13Rr^2+2r^3}{6r-R}\geq\dfrac{(4R+r)^2}{3}\iff 4R^2-7Rr-2r^2\geq 0\iff(R-2r)(4R+r)\geq 0$, which is true by Euler's Inequality. $\square$

34. **Author: r1234**

Note $\sin^2\dfrac{A}{2}=\dfrac{1-\cos A}{2}$ and then putting $\sum\cos A=1+4\cdot\prod\sin\dfrac{A}{2}$ the inequality reduces to $\prod\cos\dfrac{B-C}{2}\geq 8\cdot\prod\sin\dfrac{A}{2}$.

Using $\cos\dfrac{B-C}{2}=\dfrac{(r_a+r)}{4R\sin\frac{A}{2}}$ and $r=4R\prod\sin\dfrac{A}{2}$ the inequality reduces to $\prod(r_a+r)\geq 32Rr^2$.

We know that $r=\dfrac{\Delta}{s}$ and $r_a=\dfrac{\Delta}{s-a}$. So writing r_b,r_c and putting $R=\dfrac{abc}{4\Delta}$ the inequality reduces to $\prod(b+c)\geq 8abc$ which trivially comes from AM-GM inequality. $\square$

34. **Author: Thalesmaster**

Note that $\cos\dfrac{B-C}{2}=\dfrac{b+c}{a}\sin\dfrac{A}{2}$.

Then $\prod\cos\dfrac{B-C}{2}\geq 8\prod\sin\dfrac{A}{2}\iff\prod(b+c)\geq 8abc$, which is true according to AM-GM. $\square$

35. **Author: truongtansang89**

Let R be the radius of (O).

$$\frac{AK}{OK}+\frac{BL}{OL}+\frac{CM}{OM}\geq\frac{9}{2}\iff\frac{OK+OA}{OK}+\frac{OB+OL}{OL}+\frac{OC+OM}{OM}\geq\frac{9}{2}\iff\frac{R}{OK}+\frac{R}{OL}+\frac{R}{OM}\geq\frac{3}{2}$$

Using Ptolemy's Theorem on the cyclic quadrilateral $BOCK$,

$OB\cdot CK+OC\cdot BK=BC\cdot OK$

$$\iff\frac{R}{OK}=\frac{BC}{BK+CK}=\frac{\sin BOC}{\sin BOK+\sin COK}\iff\frac{R}{OK}=\frac{|\sin 2A|}{|\sin 2B|+|\sin 2C|}$$

Similarly, we have $\dfrac{R}{OK}+\dfrac{R}{OL}+\dfrac{R}{OM}\geq\sum\dfrac{|\sin 2A|}{|\sin 2B|+|\sin 2C|}\geq\dfrac{3}{2}$, which is Nesbitt's Inequality. $\square$

35. **Author: r1234**

Let us invert this figure w.r.t the circumcircle of $\triangle ABC$. Let AO meet the side BC at D. Define E, F similarly. Now the circumcircle of BOC is inverted to the line BC. Hence D is the inverse of K. Hence we get $AK=\dfrac{R^2\cdot AD}{OA\cdot OD}=\dfrac{R\cdot AD}{OD}$. Similarly we get $OK=\dfrac{R^2}{OD}$ Hence $\dfrac{AK}{OK}=\dfrac{AD}{R}$. Similarly $\dfrac{BL}{OL}=\dfrac{BE}{R}$ and $\dfrac{CM}{OM}=\dfrac{CF}{R}$. So now we have to prove that $\dfrac{1}{R}(AD+BE+CF)\geq\dfrac{9}{2}$.

Now let $BD : DC = x : y$, $CE : EA = y : z$ and $AF : FB = z : x$. Now using Menelaus's theorem we get $OD : OA = (x + y + z) : (y + z)$ and similar for others. Hence the inequality reduces to $(x + y + z) \cdot \left(\sum \frac{1}{y+z}\right) \geq \frac{9}{2}$ which comes from AM-GM or CS. $\square$

36. **Author: bzprules**

We have that $2s \leq 3R\sqrt{3} \implies 6s \leq 9R\sqrt{3} \implies 2rs^2\sqrt{3} \leq 9Rrs \implies 8rs^2\sqrt{3} \leq 36Rrs \implies 4(2s)\Delta\sqrt{3} \leq 36Rrs$. Since $4\Delta R = 4Rrs = abc$, we have $4(2s)\Delta\sqrt{3} \leq 9abc$.
Dividing yields $4\sqrt{3} \cdot \Delta \leq \frac{9abc}{a+b+c}$, as desired. $\square$

37. **Author: applepi2000**

Use Ravi Substitution $a = x + y, b = x + z, c = y + z$.
Then it becomes $\sum (x^2 + y^2 + 2xy)(xy + yz - xz - z^2) \geq 0$
After expanding and simplifying $\sum x^3y - 2xyz \sum x \geq 0 \iff \sum x^3y \geq 2xyz \sum x$
By Cauchy-Schwarz we have
$(x^3y + xy^3 + x^3z + xz^3 + y^3z + yz^3)(xyz^2 + xyz^2 + xy^2z + xy^2z + x^2yz + x^2yz)$
$\geq (x^2yz + x^2yz + xy^2z + x^2yz + xyz^2 + xyz^2)^2$.
Dividing by $2xyz \cdot \sum x$ gives the desired result. $\square$

37. **Author: Thalesmaster**

Lemma.
Let a, b, c be three reals and x, y, z be three nonnegative reals. The inequality $\sum x(a - b)(a - c) \geq 0$ holds if x, y, z are the side-lengths of a triangle (sufficient condition).

Proof. Use the identity $\sum x(a - b)(a - c) = \frac{1}{2} \sum (y + z - x)(b - c)^2 \geq 0$. $\square$

We have $\sum a^2b(a - b) \geq 0 \iff \sum c(a + b - c)(a - b)(a - c) \geq 0$, which is true according to the lemma, since $c(a + b - c)$, $b(c + a - b)$ and $a(b + c - a)$ are the side lengths of a triangle. $\square$

38. **Author: BigSams**

By CS, $(\sin a \cdot \sin b + \cos a \cdot \cos b) \cdot \left(\frac{\sin^3 a}{\sin b} + \frac{\cos^3 a}{\cos b}\right) \geq \left(\sin^2 a + \cos^2 a\right)^2 = 1$
$\iff \frac{\sin^3 a}{\sin b} + \frac{\cos^3 a}{\cos b} \geq \frac{1}{\sin a \cdot \sin b + \cos a \cdot \cos b} = \sec(a - b)$. $\square$

39. **Author: applepi2000**

Let's first assume that the parallelogram is not a rectangle. Then putting it on its base and straightening its slanted side will increase the height, and keep the base constant. Thus, the greatest area must be a rectangle.
Now, we must maximize ab given $2(a + b)$. By AM-GM we know this is maximized when $a = b$. Thus, the figure is a square. $\square$

40. **Author: KrazyFK**

Clearly $AC \le AB + BC$ and $AC \le CD + DA$.
We have two similar inequalities for BD and adding them we get the result.

41. **Author: xyy**

Let A_1, B_1, C_1 be the intersection of PA, PB, PB with BC, CA, AB, respectively.
We have $S = \dfrac{BL}{LC} \cdot \dfrac{CM}{MA} \cdot \dfrac{AN}{NB} = \dfrac{PC_1}{PC} \cdot \dfrac{PA_1}{PA} \cdot \dfrac{PB_1}{PB}$.
Let $x = \dfrac{PA_1}{AA_1}, y = \dfrac{PB_1}{BB_1}, z = \dfrac{PC_1}{CC_1}$.
We know that $x + y + z = \dfrac{S_{PBC}}{S_{ABC}} + \dfrac{S_{PCA}}{S_{ABC}} + \dfrac{S_{PAB}}{S_{ABC}} = 1$.
$S = \dfrac{x}{1-x} \cdot \dfrac{z}{1-z} \cdot \dfrac{z}{1-z} \le \dfrac{1}{8} \iff (x+y)(y+z)(z+x) \ge 8xyz$, which is true by AM-GM. $\square$

42. **Author: Mateescu Constantin**

The inequality rewrites as: $2R \cdot \sum \sin A \sin \dfrac{A}{2} \ \ge\ s \iff 2\sum \sin A \sin \dfrac{A}{2} \ \ge\ \sum \sin A$ $(*)$, because it is well-known that:
$\sum \sin A = \dfrac{s}{R}$. Using the substitutions $\left\| \begin{array}{c} A = \pi - 2X \\ B = \pi - 2Y \\ C = \pi - 2Z \end{array} \right\|$, where $X, Y, Z \in \left(0, \dfrac{\pi}{2}\right)$ we will transform the inequality in any triangle $(*)$ into one restricted to an acute-angled triangle. Indeed, the inequality $(*)$ is now equivalent to: $2\sum \sin 2X \cos X \ \ge\ \sum \sin 2X \iff$
$4\sum \sin X \left(1 - \sin^2 X\right) \ \ge\ \sum \sin 2X \iff 4\sum \sin X \ \ge\ 4\sum \sin^3 X + \sum \sin 2X$.
For convenience, we will denote by s, R, r the semiperimeter, circumradius and inradius respectively of the acute triangle XYZ.

Since: $\begin{cases} \sum \sin X = \dfrac{s}{R} \\[2ex] \sum \sin^3 X = \dfrac{2s(s^2 - 6Rr - 3r^2)}{8R^3} \\[2ex] \sum \sin 2X = \dfrac{2rs}{R^2} \end{cases}$

our last inequality finally becomes: $\dfrac{4s}{R} \ \ge\ \dfrac{s(s^2 - 6Rr - 3r^2)}{R^3} + \dfrac{2rs}{R^2} \iff 4R^2 + 4Rr + 3r^2 \ \ge\ s^2$, which is Gerretsen's Inequality. $\square$

43. **Author: Mateescu Constantin**

The triangle ABC is right-isosceles in C, so we can consider: $\begin{cases} AC = BC = a \\[2ex] AB = a\sqrt{2} \end{cases}$. Also, denote the ratio $\dfrac{AP}{PB} = k$, where $k > 0$.
Note that triangles ARP and PQB are right-isosceles in R and Q respectively and that:

$$\begin{cases} \dfrac{AR}{AC} = \dfrac{AP}{AB} \implies AR = a \cdot \dfrac{k}{k+1} \\ \\ \dfrac{BQ}{BC} = \dfrac{BP}{BA} \implies BQ = a \cdot \dfrac{1}{k+1} \end{cases}$$. Consequently: $$\begin{cases} [ARP] = \dfrac{a^2}{2} \cdot \dfrac{k^2}{(k+1)^2} \\ \\ [PQB] = \dfrac{a^2}{2} \cdot \dfrac{1}{(k+1)^2} \\ \\ [PQCR] = a^2 \cdot \dfrac{k}{(k+1)^2} \end{cases}$$ and since: $[ABC] = \dfrac{2a^2}{9}$, the conclusion can be restated as:

$k > 0 \implies \max\left\{\dfrac{k^2}{2(k+1)^2}, \dfrac{1}{2(k+1)^2}, \dfrac{k}{(k+1)^2}\right\} \geq \dfrac{2}{9}$, which follows from the following:

$$\begin{cases} \dfrac{k^2}{2(k+1)^2} \geq \dfrac{2}{9} \implies 5k^2 - 8k - 4 \geq 0 \implies k \geq 2 \\ \\ \dfrac{1}{2(k+1)^2} \geq \dfrac{2}{9} \implies -4k^2 - 8k + 5 \geq 0 \implies k \in \left(0, \dfrac{1}{2}\right] \qquad \square \\ \\ \dfrac{k}{(k+1)^2} \geq \dfrac{2}{9} \implies -2k^2 + 5k - 2 \geq 0 \implies k \in \left[\dfrac{1}{2}, 2\right] \end{cases}$$

44. **Author: fractals**

By the AM-GM, $\dfrac{1}{3} = \dfrac{\frac{(s-a)}{s} + \frac{(s-b)}{s} + \frac{(s-c)}{s}}{3} \geq \sqrt[3]{\dfrac{(s-a)(s-b)(s-c)}{s^3}}$.
Thus, $\dfrac{(s-a)(s-b)(s-c)}{s^3} \leq \dfrac{1}{27}$, so $s(s-a)(s-b)(s-c) \leq \dfrac{s^4}{27}$. Thus $rs = \sqrt{s(s-a)(s-b)(s-c)} \leq \dfrac{s^2}{3\sqrt{3}}$, so $\dfrac{r}{s} \leq \dfrac{1}{3\sqrt{3}}$, so $\dfrac{s}{r} \geq 3\sqrt{3}$, which is Mitrinovic's Inequality. $\square$

45. **Author: r1234**

Let AD be the median of triangle ABC which intersects the circumcircle at the point D'. Due to secant property, we get $AD \cdot DD' = \dfrac{BC^2}{4} = \dfrac{a^2}{4}$. So $DD' = \dfrac{a^2}{4m_a}$.
Now $AD' \leq 2R \iff AD + DD' \leq 2R \iff m_a + \dfrac{a^2}{4m_a} \leq 2R \iff \dfrac{4m_a^2 + a^2}{2m_a} \leq 2R$.
Now putting $m_a^2 = \dfrac{b^2+c^2}{2} - \dfrac{a^2}{4}$ we get $\dfrac{b^2+c^2}{m_a} \leq 2R$.
The cyclic summation will give us the desired result. $\square$

46. **Author: KrazyFK**

By Ptolemy's Inequality in quadrilateral $ABCE$ we have $(AB)(CE) + (BC)(AE) \geq (AC)(BE)$, and since $AB = BC$ this becomes $BC(CE + AE) \geq (AC)(BE) \iff \dfrac{BC}{BE} \geq \dfrac{AC}{CE + AE}$.
Similarly, we have $\dfrac{DE}{DA} \geq \dfrac{CE}{AE + AC}$ and $\dfrac{FA}{FC} \geq \dfrac{AE}{AC + CE}$.
Summing the three, we get $\dfrac{BC}{BE} + \dfrac{DE}{DA} + \dfrac{FA}{FC} \geq \dfrac{AC}{CE + AE} + \dfrac{CE}{AE + AC} + \dfrac{AE}{AC + CE} \geq \dfrac{3}{2}$, which is

true by Nesbitt's Inequality.
Equality holds if, and only if, all of the following conditions are true:
ACE is equilateral, $ABCE$ is cyclic, $CDEA$ is cyclic, $EFAC$ is cyclic.
From this we easily infer the congruence of ABC, CDE and EFA which tells us the hexagon is equilateral. We can also easily get that it is equiangular, and so it is regular, which is therefore the only equality case. $\square$

47. **Author: Mateescu Constantin**

We will prove that: $l_a + l_b + m_c \overset{(1)}{\leq} \sqrt{s(s-a)} + \sqrt{s(s-b)} + m_c \overset{(2)}{\leq} \sqrt{2} \cdot \sqrt{s^2 - m_c^2} + m_c \overset{(3)}{\leq} s\sqrt{3}$.
Inequality (1) follows from the well-known fact: $l_a \leq \sqrt{s(s-a)}$.
Indeed, $l_a = \dfrac{2\sqrt{bc}}{b+c} \cdot \sqrt{s(s-a)} \leq \sqrt{s(s-a)}$.

For inequality (2) let's note that: $\begin{cases} 4m_c^2 = \left(a+b+2\sqrt{(s-a)(s-b)}\right)\left(a+b-2\sqrt{(s-a)(s-b)}\right) \\ \\ a+b-2\sqrt{(s-a)(s-b)} = 2s - \left(\sqrt{s-a}+\sqrt{s-b}\right)^2 \\ \\ 2\sqrt{(s-b)(s-c)} \leq (s-a)+(s-b) = c \end{cases}$.

Whence we obtain that: $4m_c^2 \leq 2s \cdot \left(2s - \left(\sqrt{s-a}+\sqrt{s-b}\right)^2\right)$
$\implies \sqrt{s(s-a)} + \sqrt{s(s-b)} \leq \sqrt{2} \cdot \sqrt{s^2 - m_c^2}$.
The inequality (3) is clearly true since it follows from Cauchy-Schwarz Inequality, so we are done. $\square$

48. **Author: powerofzeta**

It's known that: $m_a = \dfrac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$
By CS $\sum m_a = \dfrac{1}{2}\sum\sqrt{2b^2+2c^2-a^2} \leq \dfrac{1}{2}\sqrt{3 \cdot \sum(2b^2+2c^2-a^2)} = \dfrac{3}{2}\sqrt{\sum a^2} = \dfrac{3}{2}\sqrt{2s^2 - 2r^2 - 8Rr}$
By Gerresten's Inequality, $s^2 \leq 4R^2 + 4Rr + 3r^2$
$\implies \sum m_a \leq \dfrac{3}{2}\sqrt{2(4R^2+4Rr+3r^2) - 2r^2 - 8Rr} = 3\sqrt{2R^2 + r^2}$
and by Euler's Inequality $R \geq 2r$, we get: $\sum m_a \leq 3\sqrt{2R^2 + \dfrac{R^2}{4}} = \dfrac{9}{2}R$
So it suffices to prove that $12(R-2r) + \dfrac{ab+ac+bc}{R} \geq \dfrac{18}{2}R \iff 12(R-2r) + \dfrac{s^2+r^2+4Rr}{R} \geq \dfrac{18}{2}R$
By Gerresten's inequality $s^2 + r^2 \geq 16Rr - 4r^2 \geq 14Rr$.
It suffice to prove that $12(R-2r) + \dfrac{14Rr+4Rr}{R} \geq 9R$ which is true because it's equivalent to $R \geq 2r$.
Equality holds when $R = 2r$, i.e. $\triangle ABC$ is equilateral. $\square$

48. **Author: Thalesmaster**

We use the well-known inequality $m_a + m_b + m_c \leq 4R + r$ and the identity $ab+bc+ca = s^2+r^2+4Rr$.
Then, we just have to show that: $2(4R+r) - \dfrac{s^2+r^2+4Rr}{R} \leq 12(R-2r) \iff s^2+r^2+4R^2 \geq 22Rr$.
Which immediately follows by summing up the knows results $s^2 + r^2 \geq 14Rr$ and $4R^2 \geq 8Rr$. $\square$

49. **Author: BigSams**

Lemmata.

(1) $m_a^2 = \dfrac{2b^2+2c^2-a^2}{4}$, and the cyclic versions hold as well.

(2) $\dfrac{1}{h_a^2} = \dfrac{a^2}{4S^2}$, and the cyclic versions hold as well.

$(1)\times(2) = \dfrac{m_a^2}{h_a^2} = \dfrac{a^2(2b^2+2c^2-a^2)}{16S^2} \implies a^2(2b^2+2c^2-a^2) = \dfrac{16S^2m_a^2}{h_a^2}$,
and the cyclic versions hold as well. $\square$

By Trivial Inequality, $(2a^2-b^2-c^2)^2 \geq 0$
$\iff (a^2+b^2+c^2)^2 \geq 3a^2(2b^2+2c^2-a^2) = \dfrac{3\cdot 16S^2m_a^2}{h_a^2} \iff a^2+b^2+c^2 \geq \dfrac{4\sqrt{3}Sm_a}{h_a}$.
Clearly the cyclic versions of the above result can be derived by starting with the cyclic versions of $(2a^2-b^2-c^2)^2 \geq 0$ and proceeding by the same manipulations and cyclic versions of identities, so the inequality always holds for any of $\dfrac{m_a}{h_a}, \dfrac{m_b}{h_b}, \dfrac{m_c}{h_c}$.
Thus, $a^2+b^2+c^2 \geq 4\sqrt{3}S\cdot\max\left(\dfrac{m_a}{h_a}, \dfrac{m_b}{h_b}, \dfrac{m_c}{h_c}\right)$. $\square$

50. **Author: RSM**

$AB_2 = AC_1 = b+c$, so $[AB_2C_1] = \dfrac{(b+c)^2\sin A}{2}$ and similar for others.
$[CC_1C_2] = \dfrac{c^2\sin C}{2}$. Adding up all these we get the desired result.
$[A_1A_2B_1B_2C_1C_2] = \dfrac{(a+b+c)(a^2+b^2+c^2)}{4R} + 4[ABC]$ where R is the circumradius of ΔABC and a, b, c are its sides.
Note that, $(a+b+c)(a^2+b^2+c^2) \geq 9abc$
So $[A_1A_2B_1B_2C_1C_2] \geq \dfrac{9abc}{4R} + 4[ABC] = 13[ABC]$ $\square$

51. **Author: RSM**

Note that, $r_1 = \dfrac{\Delta}{2s_{ABD}}, r_2 = \dfrac{\Delta}{2s_{ACD}}$ where s_X denotes the semi-perimeter of ΔX.
Substituting this in the inequality we get that the inequality is equivalent to
$\dfrac{s_{ABC}+m_a}{\Delta} \geq \dfrac{1}{r} + \dfrac{2}{a} \iff \dfrac{m_a}{\Delta} \geq \dfrac{2}{a} \iff \dfrac{1}{2}\cdot m_a a \geq \Delta$, which is true since $\dfrac{1}{2}\cdot m_a a \geq \dfrac{1}{2}\cdot h_a a = \Delta$. $\square$

52. **Author: Mateescu Constantin**

Right Side.

We make use of the identities: $\begin{cases} \sum \cos A = 1 + \dfrac{r}{R} \\ \\ \sum \cos B \cos C = \dfrac{s^2 + r^2 - 4R^2}{4R^2} \\ \\ \sum \sin B \sin C = \dfrac{s^2 + r^2 + 4Rr}{4R^2} \\ \\ \sum \dfrac{1}{\sin^2 \frac{A}{2}} = \dfrac{s^2 + r^2 - 8Rr}{r^2} \end{cases}$.

Thus, $8 \sum \cos A \le 9 + \sum \cos(A - B) \iff s^2 \ge 14Rr - r^2$, which is true since it is weaker than Gerretsen's Inequality: $s^2 \ge 16Rr - 5r^2$.

Left Side.

$9 + \sum \cos(A - B) \le \sum \dfrac{1}{\sin^2 \frac{A}{2}} \iff \dfrac{s^2 + r^2 + 2Rr - 2R^2}{2R^2} \le \dfrac{s^2 - 8Rr - 8r^2}{r^2}$.

Since: $\begin{cases} \dfrac{s^2 + r^2 + 2Rr - 2R^2}{2R^2} \overset{(G)}{\le} \dfrac{R^2 + 3Rr + 2r^2}{R^2} \\ \\ \dfrac{8Rr - 13r^2}{r^2} \overset{(G)}{\le} \dfrac{s^2 - 8Rr - 8r^2}{r^2} \end{cases}$

It suffices to show that: $\dfrac{R^2 + 3Rr + 2r^2}{R^2} \le \dfrac{8R - 13r}{r} \iff (R - 2r)(8R^2 + 2Rr + r^2) \ge 0$, which is true by Euler's Inequality. $\square$

53. **Author: Thalesmaster**

Using the system: $\begin{cases} a + b + c = 2s \\ ab + bc + ca = s^2 + r^2 + 4Rr \\ abc = 4sRr \end{cases}$

We have: $\dfrac{2s^4 - (a^4 + b^4 + c^4)}{[ABC]^2} \ge 38 \iff \dfrac{12s^2r^2 + 16s^2Rr - 16Rr^3 - 32R^2r^2 - 2r^4}{s^2r^2} \ge 38$

$\iff y^2(8x - 13) \ge 16x^2 + 8x + 1$, where $x = \dfrac{R}{r} \ge 2$ and $y = \dfrac{s}{r} \ge 3\sqrt{3}$.

Using Gerretsen's Inequality: $y^2 + 5 \ge 16x$, we just have to show that $(16x - 5)(8x - 13) \ge 16x^2 + 8x + 1 \iff 7x^2 - 16x + 4 \ge 0 \iff (x - 2)(7x - 2) \ge 0$

which is true by Euler's Inequality.

The value 38 is attained for an equilateral $\triangle ABC$. $\square$

54. **Author: Thalesmaster**

Using the substitutions $\begin{cases} A = \pi - 2X \\ \\ B = \pi - 2Y \\ C = \pi - 2Z \end{cases}$, for $X, Y, Z \in \left(0, \dfrac{\pi}{2}\right)$ we will transform the given inequality into an one restricted to an acute-angled triangle with side lengths x, y, z corresponding to angles X, Y, Z respectively: $\sum \sin X \le \dfrac{\sqrt{3}}{2} \cdot \sum \cos \dfrac{Y - Z}{2}$. This inequality is actually true in any triangle:

Expressing everything in terms of x, y, z using well-known formulas and then Ravi Substitution:

$$\begin{cases} x = u + v \\ y = w + u \\ z = v + w \end{cases} \iff \left(\sum \frac{u+v+2w}{\sqrt{w(u+v)}}\right)^2 \left(\sum w(u+v+2w)(u+v)\right) \geq \left(\sum u+v+2w\right)^3$$

Which is clearly true according to Hölder's Inequality. $\square$

55. **Author: gaussintraining**

By CS, $3 \cdot \sum a^2 \geq \left(\sum a\right)^2 = 4s^2 > \pi s^2 = \pi r^2 \cdot \left(\frac{s^2}{r^2}\right) = Z \cdot \left(\sum \cot \frac{A}{2}\right)^2$.

56. **Author: malcolm**

Using $AX < \max\{AB, AC\}$ for X interior to BC and similarly for the other sides we have
$AX + BY + CZ < \max\{AB, AC\} + \max\{BC, BA\} + \max\{CA, CB\} = AC + BC + BC = 2a + b$. $\square$

57. **Author: Michael Niland**

Use the following: $\begin{cases} \sum \frac{1}{a} \cos^2 \frac{A}{2} = \frac{s^2}{abc} \\ \sum \cos^2 \frac{A}{2} = 2 + \frac{r}{2R} \leq \frac{9}{4} \end{cases}$

By Chebyshev's Inequality,
$\sum \cos^4 \frac{A}{2} = \sum \left[\left(a \cos^2 \frac{A}{2}\right) \cdot \left(\frac{1}{a} \cos^2 \frac{A}{2}\right)\right]$
$\leq \frac{1}{3}\left(\cdot \sum a \cos^2 \frac{A}{2}\right) \cdot \left(\sum \frac{1}{a} \cos^2 \frac{A}{2}\right) = \frac{1}{3} \cdot \left(\sum a \cos^2 \frac{A}{2}\right) \cdot \frac{s^2}{abc}$

Again using Chebyshev's Inequality, $\sum a \cos^2 \frac{A}{2} \leq \frac{1}{3} \cdot \left(\sum a\right) \cdot \left(\sum \cos^2 \frac{A}{2}\right) \leq \frac{2s}{3} \cdot \frac{9}{4}$.

Therefore $\sum \cos^4 \frac{A}{2} \leq \frac{1}{3} \cdot \left(\frac{2s}{3} \cdot \frac{9}{4}\right) \cdot \left(\frac{s^2}{abc}\right) = \frac{s^3}{2abc}$. $\square$

58. **Author: Thalesmaster**

Using complex numbers $A(a)$, $B(b)$, $C(c)$ and $P(p)$ and the identity
$(b-c)(p-b)(p-c) + (c-a)(p-c)(p-a) + (a-b)(p-a)(p-b) = (a-b)(b-c)(c-a)$.
We have
$BC \cdot PB \cdot PC + CA \cdot PC \cdot PA + AB \cdot PA \cdot PB$
$= |(b-c)(p-b)(p-c)| + |(c-a)(p-c)(p-a)| + |(a-b)(p-a)(p-b)|$
$\geq |(b-c)(p-b)(p-c) + (c-a)(p-c)(p-a) + (a-b)(p-a)(p-b)|$
$= |(a-b)(b-c)(c-a)| = AB \cdot BC \cdot CA$
Which yields to the desired result.
Equality holds if and only if $P = H$ where H is the orthocenter of $\triangle ABC$. $\square$

59. **Author: RSM**

Suppose, PA', PB', PC' are the perpendiculars from P to the sides BC, CA, AB and $PA' = p, PB' = q, PC' = r$.
Note that $B'C' = d_A \sin A$ and similar for others.
So the inequality is equivalent to $A'B'^2 + B'C'^2 + C'A'^2 \leq 3(PA'^2 + PB'^2 + PC'^2)$
Which is true since $(PA'^2 + PB'^2 + PC'^2) = \dfrac{A'B'^2 + B'C'^2 + C'A'^2}{3} + 3PG^2$ where is G is the centroid of $A'B'C'$.
Equality holds when P and G coincides, i.e. when P is the symmedian point of ABC. $\square$

60. **Author: Thalesmaster**

Using the condition , we have $(b \geq c \text{ or } c > b) \implies (b > a \text{ or } c > a)$.
In the two cases, a is not the greatest side, so $A < \dfrac{\pi}{2}$ We want to show that: $\angle BAC < \dfrac{\angle ABC + \angle ACB}{2}$ $\Leftrightarrow A < \dfrac{\pi}{3}$ We have: $a < \dfrac{b+c}{2} \iff \dfrac{a}{R} < \dfrac{b}{2R} + \dfrac{c}{2R} \iff 2\sin A < \sin B + \sin C$ $\iff 3\sin A < \sum \sin A = \dfrac{s}{R} \leq \dfrac{3\sqrt{3}}{2}$ So: $\sin A \leq \dfrac{\sqrt{3}}{2} = \sin\dfrac{\pi}{3}$ The function sin is increasing on the interval $]0; \dfrac{\pi}{2}[$. Hence $A \leq \dfrac{\pi}{3}$ since we proved that $A < \dfrac{\pi}{2}$. $\square$

61. **Author: Mateescu Constantin**

By squaring both sides of this inequality and taking into account the identity: $m_a^2 + m_b^2 + m_c^2 = \dfrac{3(a^2+b^2+c^2)}{4}$, we are left to prove that: $\sum m_b m_c \leq \dfrac{1}{2}\sum a^2 + \dfrac{1}{4}\sum bc$, which follows by summing up the inequalities: $m_b m_c \leq \dfrac{a^2}{2} + \dfrac{bc}{4}$ a.s.o. Indeed, $m_b m_c \leq \dfrac{a^2}{2} + \dfrac{bc}{4} \iff 16m_b^2 m_c^2 \leq \left(2a^2 + bc\right)^2 \iff 16 \cdot \dfrac{2(c^2+a^2) - b^2}{4} \cdot \dfrac{2(a^2+b^2) - c^2}{4} \leq \left(2a^2 + bc\right)^2 \iff (b-c)^2(a+b+c)$ $(a-b-c) \leq 0$, which is true from the Trivial and Triangle Inequslities. $\square$

(**BigSams** used the same method in his submission to the Mathematical Reflections bi-monthly journal, where the problem was originally from)

62. **Author: Thalesmaster**

The inequality is equivalent to $\sum \cos\dfrac{A}{2} \geq \dfrac{\sqrt{2}}{2} + \sqrt{\dfrac{1}{2}} + 2(3\sqrt{3} - 2\sqrt{2})\prod \cos\dfrac{A}{2}$

Use the substitution: $\begin{cases} X = \dfrac{\pi - A}{2} \\ Y = \dfrac{\pi - B}{2} \\ Z = \dfrac{\pi - C}{2} \end{cases}$

Denote s, R, r the semi-perimeter, the circumradius and the inradius of acute $\triangle XYZ$, then the desired inequality is equivalent to:
$\iff \sum \sin X \geq \dfrac{\sqrt{2}}{2} + \sqrt{\dfrac{1}{2}} + 2(3\sqrt{3} - 2\sqrt{2})\prod \sin X$
$\iff s \geq \sqrt{2}R + (3\sqrt{3} - 2\sqrt{2})r$

$\iff s^2 \geq 2R^2 + (6\sqrt{6} - 8)Rr + (35 - 12\sqrt{6})r^2$
Using Walker's Inequality: $s^2 \geq 2R^2 + 8Rr + 3r^2$ (since $\triangle XYZ$ is acute-angled), we just have to show that:
$2R^2 + 8Rr + 3r^2 \geq 2R^2 + (6\sqrt{6} - 8)Rr + (35 - 12\sqrt{6})r^2$
$\Leftrightarrow (16 - 6\sqrt{6})Rr \geq 2(16 - 6\sqrt{6})r^2$
$\Leftrightarrow R \geq 2r$, which is Euler's Inequality. $\square$

63. **Author: Mateescu Constantin**

Lemma. Let ABC be a triangle and let D be a point on the side $[BC]$ so that: $\dfrac{BD}{DC} = k$, $k > 0$. Then: $\dfrac{c^2 + kb^2}{\sqrt{(1+k)(c^2 + kb^2) - ka^2}} \leq 2R$.

Proof. Using the dot product, one can show the distance: $AD^2 = \dfrac{c^2 + kb^2}{1+k} - \dfrac{ka^2}{(1+k)^2}(*)$.
Let w be the circumcircle of $\triangle ABC$ and let $\{X\} = AD \cap w$.

$$\text{Thus, } \left\{\begin{array}{c} AD \cdot DX = BD \cdot CD \\ BD = \dfrac{ka}{1+k} \; ; \; CD = \dfrac{a}{1+k} \end{array}\right\| \implies AD \cdot DX = \frac{ka^2}{(1+k)^2} \implies DX = \frac{ka^2}{(1+k)^2 \cdot AD}.$$

Moreover, since AX is a chord in the circle w, it follows that: $AX \leq 2R \iff AD + DX \leq 2R \iff$ $AD + \dfrac{ka^2}{(1+k)^2 \cdot AD} \leq 2R \iff$

$$\iff (1+k)^2 \cdot AD^2 + ka^2 \leq 2R \cdot AD \cdot (1+k)^2 \overset{(*)}{\iff} c^2 + k \cdot b^2 \leq 2R \cdot AD \cdot (1+k)$$

$$\iff \frac{c^2 + kb^2}{\sqrt{(1+k)(c^2 + kb^2) - ka^2}} \leq 2R,$$ which is exactly what we wanted to prove. $\square$

Particularly, for $k = \dfrac{a^2}{b^2}$ in the previous lemma we obtain: $\dfrac{b(c^2 + a^2)}{\sqrt{a^2b^2 + b^2c^2 + c^2a^2}} \leq 2R$ and making use of the well-known relation $R = \dfrac{abc}{4\Delta}$, our last inequality simplifies to: $\dfrac{c}{a} + \dfrac{a}{c} \leq \dfrac{\sqrt{a^2b^2 + b^2c^2 + c^2a^2}}{2\Delta}$.
In a similar manner we can prove the analogous inequalities, therefore solving the problem. $\square$

64. **Author: Mateescu Constantin**

It will be shown that: $\Delta \overset{(1)}{\geq} r \cdot \sqrt{\dfrac{1}{3} \cdot \sum m_b m_c + \dfrac{1}{2} \cdot \sum bc} \overset{(2)}{\geq} r \cdot \sqrt{\dfrac{2}{3} \cdot \sum m_b m_c + r(4R + r)}$

Proof of Inequality (1)
Taking into account the known identities: $\Delta = r \cdot s$ and $\sum bc = s^2 + r^2 + 4Rr$ our inequality is succesively equivalent to: $s \geq \sqrt{\dfrac{1}{3} \cdot \sum m_b m_c + \dfrac{1}{2} \cdot \sum bc}$

$$\iff s^2 \geq \frac{1}{3} \cdot \sum m_b m_c + \frac{1}{2} \cdot \left(s^2 + r^2 + 4Rr\right) \iff \frac{s^2 - 4Rr - r^2}{2} \geq \frac{1}{3} \cdot \sum m_b m_c$$

$$\iff \frac{3(a^2 + b^2 + c^2)}{4} \geq \sum m_b m_c \iff \sum m_a^2 \geq \sum m_b m_c,$$ which is obviously true. $\square$

Proof of Inequality (2)
Squaring both sides of this inequality, we are left to show that:
$2\sum m_b m_c + 3(s^2 + r^2 + 4Rr) \geq 4\sum m_b m_c + 6r(4R + r)$
$\iff 3(s^2 - 4Rr - r^2) \geq 2\sum m_b m_c \iff \sum m_a^2 \geq \sum m_b m_c$, which is clearly true. $\square$

65. **Author: BigSams**

Problem Rewording. In pentagon $ABCDE$, prove that:

$$(AC+BE)AB+(BD+CA)BC+(CE+DB)CD+(DA+EC)DE+(EB+AD)EA$$
$$> AC^2+BD^2+CE^2+DA^2+EB^2$$

Solution. By Triangle Inequality, $AB+BC>CA \implies (AB+BC)AC>AC^2$.
Repeating with $\triangle BCD, \triangle CDE, \triangle DEA, \triangle EAB$ and summing all five yields the result. $\square$

66. **Author: gaussintraining**

Since $l_a = \dfrac{2bc}{b+c}\cos\dfrac{A}{2} = \dfrac{2\sqrt{bc}}{b+c}\sqrt{s(s-a)} \le \sqrt{s(s-a)}$ by AM-GM, it follows that $l_a^2 \le s(s-a)$.
The analogous relationships also hold, yielding $\sum l_a^2 \le 3s^2-(a+b+c)s = s^2$. $\square$

67. **Author: jatin**

Let E and F be the midpoints of AC and BD respectively. We know R is the midpoint of EF. Note that E and F lie on the circle with diameter OP. And hence $OP \ge OE$ as well as $OP \ge OF$. Now, OR is a median of $\triangle OEF$. Therefore, $OR \le OF$ or $OR \le OE$. Hence, $OP \ge OR$. $\square$

68. **Author: Mateescu Constantin**

Problem 61 from this marathon was equivalent to: $\sum m_b m_c \le \dfrac{1}{2}\sum a^2 + \dfrac{1}{4}\sum bc$. Thus we are left to prove that: $\dfrac{1}{2}\sum a^2 < \sum bc$ which is obviously true, since it rewrites as: $2(s^2-r^2-4Rr) < 2(s^2+r^2+4Rr) \iff 0 < r^2+4Rr$. $\square$

Note. BigSams commented afterwards that a more elementary final step is by Triangle Inequality, $\sum a(b+c-a) > 0 \iff 2\cdot\sum ab > \sum a^2$.

69. **Author: Mateescu Constantin**

Problem Rewording.
Let ABC be a triangle and let $M \in [AC]$, $N \in [BC]$, $L \in [MN]$.

Prove that the following inequality holds: $\boxed{\sqrt[3]{S} \ge \sqrt[3]{S_1}+\sqrt[3]{S_2}}$, where $\left\|\begin{array}{c} S=[ABC] \\ S_1=[AML] \\ S_2=[BNL] \end{array}\right\|$.

Soliution.
It is obvious that the given inequality holds when at least one of the points M, N or L coincide with one of the end points of the segments they lie on. Also, note that in such cases equality is attained when either $A=M=L$ and $C=N$ OR $B=N=L$ and $C=M$. Now we will draw our attention to the case in which $M \in (AC)$, $N \in (BC)$ and $L \in (MN)$. Let us consider $\dfrac{AM}{MC}=k$, $\dfrac{BN}{NC}=q$,

$\dfrac{ML}{LN} = r$, where $k, q, r > 0$. Therefore,

$$\left\| \begin{array}{lllll} \dfrac{AM}{MC} = k & \implies & \dfrac{[AML]}{[CML]} = k & \implies & S_1 = k \cdot [CML] \\ \dfrac{ML}{LN} = r & \implies & \dfrac{[CML]}{[CNL]} = r & \implies & [CML] = \dfrac{r}{r+1} \cdot [MNC] \\ \dfrac{BN}{NC} = q & \implies & \dfrac{[BMN]}{[MNC]} = q & \implies & [MNC] = \dfrac{1}{q+1} \cdot [BMC] \\ \dfrac{AM}{MC} = k & \implies & \dfrac{[BMA]}{[BMC]} = k & \implies & [BMC] = \dfrac{1}{k+1} \cdot S \end{array} \right\|$$

$$\implies S_1 = \frac{kr}{(k+1)(q+1)(r+1)} \cdot S$$

$$\left\| \begin{array}{lllll} \dfrac{BN}{NC} = q & \implies & \dfrac{[BNL]}{[CNL]} = q & \implies & S_2 = q \cdot [CNL] \\ \dfrac{ML}{LN} = r & \implies & \dfrac{[CML]}{[CNL]} = r & \implies & [CNL] = \dfrac{1}{r+1} \cdot [MNC] \\ \dfrac{BN}{NC} = q & \implies & \dfrac{[BMN]}{[MNC]} = q & \implies & [MNC] = \dfrac{1}{q+1} \cdot [BMC] \\ \dfrac{AM}{MC} = k & \implies & \dfrac{[BMA]}{[BMC]} = k & \implies & [BMC] = \dfrac{1}{k+1} \cdot S \end{array} \right\|$$

$$\implies S_2 = \frac{q}{(k+1)(q+1)(r+1)} \cdot S$$

Consequently, the proposed inequality reduces to:
$\sqrt[3]{S} \ \geq \ \sqrt[3]{\dfrac{kr}{(k+1)(q+1)(r+1)} \cdot S} + \sqrt[3]{\dfrac{q}{(k+1)(q+1)(r+1)} \cdot S} \quad \iff \quad \sqrt[3]{(k+1)(q+1)(r+1)} \ \geq$ $\sqrt[3]{kr} + \sqrt[3]{q}$.

Taking $k = x^3$, $r = y^3$ and $q = z^3$, where $x, y, z > 0$ it suffices to show that:
$\left(x^3+1\right)\left(y^3+1\right)\left(z^3+1\right) \ \geq \ (xy+z)^3 \iff x^3y^3z^3 + x^3z^3 + y^3z^3 + x^3 + y^3 + 1 \ \geq \ 3x^2y^2z + 3xyz^2$,
which follows by adding the following two inequalities obtained from AM-GM inequality:

$$\begin{cases} x^3y^3z^3 + x^3 + y^3 \ \geq \ 3x^2y^2z \\ \\ x^3z^3 + y^3z^3 + 1 \ \geq \ 3xyz^2 \end{cases}.$$

In this case, equality occurs iff $x = y = z = 1$, in other words, when the points M, N and L are the midpoints of the segments $[AC]$, $[BC]$ and $[MN]$ respectively. $\square$

70. **Author: Goutham**

Let P_1 be the symmetric of point P w.r.t. the midpoint of side $[BC]$. Define P_2 and P_3 in a similar manner.
By Ptolemy's Theorem, for a convex quadrilater $MNPQ$, $MN \cdot PQ + NP \cdot MQ \geq 2[MNPQ]$, with equality if and only if $MNPQ$ is cyclic and $MP \perp NQ$.
Applying this to convex quadrilaterals ABP_1C, BCP_2A, CAP_3B, we get:

$$\begin{cases} b \cdot PC + c \cdot PB \geq 2(\Delta + [P_1BC]) \\ \\ a \cdot PC + c \cdot PA \geq 2(\Delta + [P_2CA]) \\ \\ a \cdot PB + b \cdot PA \geq 2(\Delta + [P_3AB]) \end{cases}$$

Adding them gives that $LHS \geq 2(3\Delta + [P_1BC] + [P_2AC] + [P_3AB])$ for which we use $[P_1BC] = [PBC]$ and so on to get that $LHS \geq 8\Delta = RHS$. $\square$

71. **Author: Mateescu Constantin**

Let us denote $\dfrac{AP}{PC} = k$, where $k > 0$. Thus, $\begin{cases} AP = \dfrac{k}{k+1} \cdot b \\ \\ PC = \dfrac{1}{k+1} \cdot b \end{cases}$

By Pythagoras' theorem, applied in $\triangle PBC$ one obtains: $PB = \sqrt{a^2 + \dfrac{b^2}{(k+1)^2}}$. Hence, we are left to show that: $\dfrac{c - \sqrt{a^2 + \frac{b^2}{(k+1)^2}}}{\frac{k}{k+1} \cdot b} > \dfrac{c-a}{b} \iff c - \sqrt{a^2 + \dfrac{b^2}{(k+1)^2}} > \dfrac{k}{k+1} \cdot (c-a) \iff$
$\iff \dfrac{c+ak}{k+1} > \sqrt{a^2 + \dfrac{b^2}{(k+1)^2}} \iff (c+ak)^2 > a^2(k+1)^2 + b^2 \iff$
$\iff c^2 + 2ack + a^2k^2 > a^2k^2 + 2a^2k + a^2 + b^2 \overset{c^2=a^2+b^2}{\iff} c > a$, which is true. $\square$

72. **Author: Mateescu Constantin**

Construct the lines passing through the vertices of triangle ABC so that they are parallel to the sides BC, CA and AB respectively. The intersection of these three lines determines a new triangle $A'B'C'$, where A is the midpoint of segment $B'C'$. Thus, $AP = BC = AB' = AC'$, so $\widehat{B'PC'} = 90^\circ$. Now it follows that: $\widehat{A'PC'} + \widehat{B'PA'} = 270^\circ$, wherefrom one has either $\widehat{B'PA'} \leq 135^\circ$ or $\widehat{A'PC'} \leq 135^\circ$. Let us consider the first case. By denoting $x = PB'$, $y = PA'$, $2c = A'B'$ and using the Law of Cosines in triangle $B'PA'$ we obtain:

$$4c^2 = x^2 + y^2 - 2xy \cdot \cos\left(\widehat{B'PA'}\right) \leq x^2 + y^2 + 2xy \cdot \frac{\sqrt{2}}{2} \leq \left(x^2 + y^2\right)\left(1 + \frac{\sqrt{2}}{2}\right) \quad (*)$$

Moreover, by the theorem of median applied in triangle $B'PA'$ we get:

$$CP^2 = \frac{2(x^2+y^2) - 4c^2}{4} \overset{(*)}{\geq} \frac{1}{4}\left(2 \cdot \frac{4c^2}{1 + \frac{\sqrt{2}}{2}} - 4c^2\right) = \left[\left(\sqrt{2} - 1\right) \cdot AB\right]^2$$

which implies $\dfrac{CP}{AB} \geq \sqrt{2}-1$. Equality occurs when $x=y$ and $\widehat{A'PC'} = \widehat{B'PA'} = 135°$, so when $A = 45°$, $B = C = 67.5°$ and P is the orthocenter of triangle ABC. $\square$

73. **Author: Mateescu Constantin**

Using the identities: $\begin{cases} \sum a^2(s-b)(s-c) = 4s^2r(R-r) \\ \\ (s-a)(s-b)(s-c) = sr^2 \end{cases}$ the given inequality is equivalent to:

$\dfrac{\sum a^2(s-b)(s-c)}{\prod(s-a)} \geq 6R\sqrt{3} \iff \dfrac{4s^2r(R-r)}{sr^2} \geq 6R\sqrt{3} \iff s \geq \dfrac{3Rr\sqrt{3}}{2(R-r)}$

We will now show that this inequality is weaker than the known Gerretsen $s^2 \geq 16Rr - 5r^2$.

Indeed, by squaring both sides of our previous inequality, it suffices to prove that:

$16Rr - 5r^2 \geq \dfrac{27R^2r^2}{4(R-r)^2} \iff 4(R-r)^2(16Rr-5r^2) \geq 27R^2r^2 \iff r(R-2r)\left(64R^2 - 47Rr + 10r^2\right) \geq 0$, which is obviously true since $R \geq 2r$ (Euler).

Equality is attained if and only if $\triangle ABC$ is equilateral. $\square$

Remark. Here is a sketch of obtaining the first mentioned identity. Since $(s-b)(s-c) = bc - s(s-a)$, we get: $\sum a^2(s-b)(s-c) = \sum a^2\left[bc - s(s-a)\right] = abc\sum a - s^2\sum a^2 + s\sum a^3$, and further one has to use the well known identities: $\begin{cases} a^2+b^2+c^2 = 2(s^2-r^2-4Rr) \\ \\ a^3+b^3+c^3 = 2s(s^2-6Rr-3r^2) \end{cases}$.

74. **Author: BigSams**

In an arbitrary regular polygon X, let the inradius be r and the sidelength be s.

Note that the perimeter of X is always greater than the circumference of the incircle.

$\implies sn > 2\pi r \iff \dfrac{n}{r} > \dfrac{2\pi}{s}$.

Also note that $[X] = \dfrac{s \cdot \sum_{i=1}^n x_i}{2} = n \cdot \dfrac{sr}{2} \implies \displaystyle\sum_{i=1}^n x_i = nr$.

By CS, $\displaystyle\sum_{i=1}^n \frac{1}{x_i} \geq \frac{n^2}{\sum_{i=1}^n x_i} = \frac{n^2}{nr} = \frac{n}{r}$. Thus, $\displaystyle\sum_{i=1}^n \frac{1}{x_i} > \frac{2\pi}{s}$. $\square$

75. **Author: jatin**

Lemma.

The vertex of an angle α is at O. A is a fixed point inside the acute angle. On the sides of the angle, points M and N are taken such that $\angle MAN = \beta$ where $\alpha+\beta < \pi$. Then the area of the quadrilateral $OMAN$ reaches its maximum when $AM = AN$.

Proof.

Let M, N be points satisfying the given conditions such that $AM = AN$. Let M', N' be any [b]other[/b] points satisfying the given conditions.

Then we will prove that $[OM'AN'] < [OMAN]$. Now, $\angle M'AN' = \beta, \angle AM'M = 2\pi - \alpha - \beta - \angle ON'A > \pi - \angle ON'A = \angle AN'N$. Also, $\angle MAM' = \angle NAN'$ and hence $M'A < N'A$.

Thus, $[M'AM] < [N'AN] \Rightarrow [OM'AN'] < [OMAN]$. $\square$

So we have to find out on what conditions we can find on the sides on the sides of the angle points M and N such that $\angle MAN = \phi$ and $MA = AN$. Circumscribe a circle about the triangle MON. Since $\alpha + \beta + \phi < \pi$, the point A is located outside the circle. If L is the point of intersection of OA and the circle, then: $\angle AMN = \dfrac{\pi - \phi}{2} > \angle LMN = \angle LON$ and $\angle ANM = \dfrac{\pi - \phi}{2} > \angle LOM$. Thus, if $\alpha, \beta < \dfrac{\pi - \phi}{2}$, then it is possible to find points M and N such that $MA = AN$ and $\angle MAN = \phi$. If the conditions are not fulfilled then such points cannot be found. In this case, the quadrilateral of maximal area degenerates into a triangle (either M or N coincides with O). $\square$

76. **Author: dr_Civot**

Take $a = b = c$ to get that $k > 1$.
Let $a = x + y$, $b = y + z$, $c = z + x$ by Ravi Transformation.
The inequality becomes $3k \sum xy + k \sum x^2 > 2 \sum x^2 + 2 \sum xy$.
$k = 2$ works because by Triangle Inequality $\sum a(b + c - a) > 0 \iff 2 \cdot \sum ab > \sum a^2$, so $k \leq 2$.
Suppose that there exists a $1 < k < 2$ which works. Take $x = \sqrt{\dfrac{A}{2-k}}, y = z = \dfrac{1}{x}$.
The inequality becomes $LHS = (3k - 2) \sum xy > (2 - k) \sum x^2 = RHS$.
It will be shown that there is value of A for each $1 < k < 2$ such that $RHS - LHS > 0$, which will mean that $1 < k < 2$ does not exist work.
$RHS > (2 - k)x^2 = A$
$LHS = \dfrac{A(6k - 4) + (2 - k)(3k - 2)}{A}$
$RHS - LHS > 0 \iff A^2 - A(6k - 4) + (k - 2)(3k - 2) > 0$, which is true for sufficiently large A. $\square$

77. **Author: applepi2000**

Let $ad_a = x, bd_b = y, cd_c = z$.
Then from triangles MAB, MAC, MBC we have $\dfrac{1}{2}(x + y + z) = S \implies 2\Delta = x + y + z$.
We need to show $xy + yz + zx \leq \dfrac{4\Delta^2}{3}$. But this is true by Cauchy-Schwarz:
$xy + yz + zx \leq \dfrac{1}{3}(x + y + z)^2 = \dfrac{4}{3}\Delta^2$ and we are done. Equality holds iff $x = y = z$, i.e. $M = G$. $\square$

78. **Author: dr_Civot**

A power of point I is $P(I) = AI \cdot IX = OI^2 - R^2 = 2rR$, so $IX = \dfrac{2rR}{AI}$.
Hence, inequality becomes $8r^3R^3 \geq (AI \cdot BI \cdot CI)^2$.
On the other hand $r = \dfrac{\Delta}{s}$ and $R = \dfrac{abc}{4\Delta}$, so $rR = \dfrac{abc}{4s}$.
Let $a = x + y, b = y + z, c = z + x$, where x, y, z are segments that incircle divide sides of triangle.
Then $rR = \dfrac{(x + y)(y + z)(z + x)}{4(x + y + z)}$.
$AI^2 = x^2 + r^2 = x^2 + \dfrac{P^2}{s^2} = x^2 + \dfrac{xyz}{(x + y + z)}$. Now inequality becomes

$((x+y)(y+z)(z+x))^3 \geq 8(x^2(x+y+z)+xyz)(y^2(x+y+z)+xyz)(z^2(x+y+z)+xyz)$.
But we have $x^2(x+y+z)+xyz = x(x+y)(x+z)$, so our inequality is equivalent to $(x+y)(y+z)(z+x) \geq 8xyz$, which is true by AM-GM. $\square$

79. **Author: applepi2000**

Say without loss of generality $a \geq b \geq c > 0$, since the inequality is symmetric.
Multiplying the given by abc gives $\sum c(a^2+b^2-c^2) > 2abc \iff \sum a^2b + \sum a^2c > \sum a^3 + 2abc$
Now, use the identity $(a+b-c)(a-b+c)(-a+b+c) = \sum a^2b - \sum a^3 - 2abc$.
Then the given is $(a+b-c)(a-b+c)(-a+b+c) > 0$.
Now note that $a+(b-c) \geq a > 0$ and $(a-b)+c \geq c > 0$, this becomes $-a+b+c > 0 \iff b+c > a$
Also, rearranging the two strict inequalities above gives $a+b > c$ and $a+c > b$. Thus, a, b, c are sides of a triangle. $\square$

80. **Author: dr_Civot**

If $\angle B = \angle C$ then it's clear that $AP = AQ$.
Now assume that $\angle B < \angle C$. Then $\angle APB > 90$. Let M be midpoint of BC, then is $B-M-P$ [$*$].
$CP = BQ$ and $CM = BM \implies MP = MQ$, but that is possible just if $Q-M-P$ [$**$].
[$*$], [$**$], $Q \in [BC] \implies B-Q-P. \implies$ In triangle AQP $\angle QPA > 90 > \angle PQA$ so $AQ > AP$. $\square$

80. **Author: Mateescu Constantin**

If D is a point belonging to the segment $[BC]$ and $\dfrac{BD}{DC} = k \in \mathbb{R}_+$ then: $AD^2 = \dfrac{c^2+kb^2}{1+k} - \dfrac{ka^2}{(1+k)^2}$
(this can be easily proved by using the dot product i.e. $AD^2 = \overrightarrow{AD} \cdot \overrightarrow{AD}$, where $\overrightarrow{AD} = \dfrac{\overrightarrow{AB} + k \cdot \overrightarrow{AC}}{1+k}$ a.s.o.)
Returning to our problem, let's observe that: $\dfrac{BP}{PC} = \dfrac{b}{c}$ (by Angle Bisector Theorem) and $\dfrac{BQ}{QC} = \dfrac{PC}{BP} = \dfrac{c}{b}$, whence, by using the previous relation for $D \in \{P, Q\}$ one has: $\begin{cases} AP^2 = bc - \dfrac{a^2bc}{(b+c)^2} \\ AQ^2 = \dfrac{b^3+c^3}{b+c} - \dfrac{a^2bc}{(b+c)^2} \end{cases}$
(also note that the first equality can be derived from the known identity $AP = \dfrac{2bc}{b+c}\cos\dfrac{A}{2}$ - the length of the internal bisector drawn from vertex A).
Thus, $AQ \geq AP \iff \dfrac{b^3+c^3}{b+c} \geq bc \iff b^2 - bc + c^2 \geq bc \iff (b-c)^2 \geq 0$, which is true. $\square$

81. **Author: Mateescu Constantin**

Using the identities: $\prod l_a = \dfrac{16Rr^2s^2}{s^2+r^2+2Rr}$ and $\Delta = r \cdot s$ the given inequality reduces to:
$\dfrac{16Rr^2s^2}{s^2+r^2+2Rr} \leq \dfrac{r^2s^2}{r} \iff 16Rr \leq s^2+r^2+2Rr \iff s^2 \geq 14Rr - r^2$, which is weaker than the well known Gerretsen's Inequality $s^2 \geq 16Rr - 5r^2$.

Indeed $16Rr - 5r^2 \geq 14Rr - r^2 \iff 2Rr \geq 4r^2 \iff R \geq 2r \iff$ Euler's Inequality. $\square$

Remark. The first mentioned identity can be proved like this:
$\prod l_a = \prod \frac{2bc}{b+c} \cos\frac{A}{2} = \frac{8a^2b^2c^2 \prod \cos\frac{A}{2}}{(a+b+c)(ab+bc+ca) - abc} = \frac{16Rr^2s^2}{s^2+r^2+2Rr}$

82. **Author: gaussintraining**

Left Side.
Since $\sum a^2 = 2s^2 - 2r^2 - 8Rr$, the inequality is equivalent to $2s^2 \leq 2r^2 + 8Rr + 9R^2$. By comparison to Gerretsen's Inequality i.e. $s^2 \leq 4R^2 + 4Rr + 3r^2$, we see that it is weaker since $9R^2 + 8Rr + 2r^2 \geq 8R^2 + 8Rr + 6r^2 \implies R^2 \geq 4r^2$, which follows from Euler's Inequality. $\square$

Right Side.
Again, since $\sum a^2 = 2s^2 - 2r^2 - 8Rr$, the inequality is equivalent to $s^2 \geq r^2 + 13Rr$. Again, by comparison to Gerretsen's Inequality i.e $s^2 \geq 16Rr - 5r^2$, we see that it is weaker since $16Rr - 5r^2 \geq r^2 + 13Rr \implies 3Rr \geq 6r^2$, which again follows from Euler's Inequality. $\square$

83. **Author: r1234**

We prove it using complex numbers. Let z_1,z_2,z_3 be the three vertices of the triangle ABC.
Now we consider the function $g(z) = \sum \frac{(z-z_1)(z-z_2)}{(z_3-z_1)(z_3-z_2)}$.
We see that $g(z_1) = g(z_2) = g(z_3) = 1$. Since this a two degree polynomial so we conclude that $g(z) = 1$.
So $1 = g(z) \leq \sum \frac{|z-z_1||z-z_2|}{|z_3-z_1||z_3-z_2|} = \sum \frac{DA \cdot DB}{BC \cdot CA}$ and hence the result follows.
It can be checked that the equality holds when D is the orthocenter. $\square$

84. **Author: Mateescu Constantin**

Note that $\triangle\, I_aI_bI_c$ is acute-angled and I is its orthocenter. Thus, $II_a = 2R_{\triangle I_aI_bI_c} \cos\left(\widehat{I_bI_aI_c}\right)$ and since $R_{\triangle\, I_aI_bI_c} = 2R$ and $\angle I_a = 90^\circ - \frac{A}{2}$ we obtain: $II_a = 4R\sin\frac{A}{2}$. The proposed inequality is now equivalent to: $64R^3 \cdot \frac{r}{4R} \leq 8R^3 \iff 2r \leq R$, which is Euler's Inequality. $\square$

Remark. The identity $R_{\triangle\, I_aI_bI_c} = 2R$ can be easily derived. Since $I_bI_c = 4R\cos\frac{A}{2}$ and by using the law of sines one gets: $R_{\triangle\, I_aI_bI_c} = \frac{I_bI_c}{2\sin I_a} = \frac{4R\cos\frac{A}{2}}{2\sin\left(90^\circ - \frac{A}{2}\right)} = 2R$.

85. **Author: crazyfehmy**

The inequality is equivalent to $(\cos A + \cos B + \cos C)(\cot A + \cot B + \cot C) \geq \frac{3\sqrt{3}}{2}$, where A, B, C are angles of an acute triangle.

The function $f(x) = \dfrac{\cos x}{\sqrt{\sin x}}$ is concave upward for $0 < x < \dfrac{\pi}{2}$ and therefore we are done using Cauchy-Schwarz and Jensen inequality. $\square$

86. **Author: KingSmasher3**

Left Side.
For the left hand side of the problem, we have $(a+b+c)(ab+ac+bc) = a^2b+a^2c+ab^2+b^2c+ac^2+bc^2+3abc$. By Schur's Inequality, RHS $\le a^3+b^3+c^3+3abc+3abc = a^3+b^3+c^3+6abc$. $\square$

Right Side.
For the right hand side of the problem, we use the fact that a, b, c are the sides of a triangle, so we let $a = x+y, b = x+z, c = y+z$.
Thus the inequality becomes $(3,0,0)+8(2,1,0)+18xyz > (3,0,0)+8(2,1,0)+10xyz$, which is clearly true since $x, y, z > 0$. $\square$

87. **Author: applepi2000**

Assuming $F = D$. Then it is equivalent with $4(ED)^2 \ge (BC)^2$.
Let $AD = a, AE = b$. Then by Law of Cosines, $(ED)^2 = a^2+b^2-2ab\cos A$.
$(BC)^2 = 2(a+b)^2 - 2(a+b)^2\cos A$
Now note that we need $4(ED)^2 - (BC)^2 \ge 0$.
Or, in other words $2a^2+2b^2-4ab+(2a^2+2b^2-4ab)\cos A \ge 0$.
$2(a-b)^2(1+\cos A) \ge 0$. This is true since $\cos A > -1$. For equality to hold, we must have $a = b$, or D, E are the midpoints of AB, AC respectively. $\square$

88. **Author: chronondecay**

First assume that the triangle has an obtuse angle at A. It is well-known that A is also the orthocentre of HBC, which is an acute triangle. Thus we have $BH \ge BA, CA \le CH$ since $\angle HAB, \angle HAC$ are obtuse. Thus we may swap H and A, and the LHS of the inequality decreases.

Now assume that ABC is non-obtuse.
Let the feet of altitudes from A, B be A', B' respectively. Then
$AA' = \dfrac{2[ABC]}{BC} = \dfrac{AB \cdot AC \cdot \sin A}{BC}$, $AB' = AC\cos A$, $AH \cdot AA' = AB \cdot AB' \implies \dfrac{AH}{BC} = \cot A$.
Finally by Jensen's Inequality on $\cot x$, which is concave up on $\left[0, \dfrac{\pi}{2}\right)$, we get
$\sum \cot A \ge 3\cot \dfrac{\sum A}{3} = 3\cot\dfrac{\pi}{3} = 3\sqrt{3}$.
Equality occurs iff $A = B = C = \dfrac{\pi}{3}$, ie. when $\triangle ABC$ is equilateral. $\square$

89. **Author: gold46**

Consider inversion with respect to A_1 with power 1. Let A_i' be image of A_i.
Applying triangle inequality, we have $A_1'A_n' \le A_1'A_2' + \cdots + A_{n-1}'A_n'$
$$\implies A_1A_n\left(\frac{1}{MA_1 \cdot MA_2} + \frac{1}{MA_2 \cdot MA_3} + \cdots + \frac{1}{MA_{n-1} \cdot MA_n}\right) \ge \frac{A_1A_n}{MA_1 \cdot MA_n}$$
$$\implies \frac{1}{MA_1 \cdot MA_2} + \frac{1}{MA_2 \cdot MA_3} + \cdots + \frac{1}{MA_{n-1} \cdot MA_n} \ge \frac{1}{MA_1 \cdot MA_n}$$ as desired. $\square$

90. **Author: Mateescu Constantin**

It is well-known that: $3 \cdot \left(QA^2 + QB^2 + QC^2\right) = 9 \cdot QG^2 + \left(a^2 + b^2 + c^2\right)$. Therefore, $QA^2 + QB^2 + QC^2 \geq \frac{1}{3} \cdot \left(a^2 + b^2 + c^2\right)$, so the minimum is $\frac{a^2 + b^2 + c^2}{3}$, which is attained for $Q = G$. $\square$

91. **Author: BigSams**

Let $\triangle_m$ be the median with side lengths equal to the medians of $\triangle$.
Applying the reverse Hadwiger-Finsler Inequality to $\triangle_m$,
$\sum m_a^2 \leq 4\sqrt{3}S_m + 3 \cdot \sum (m_a - m_b)^2 = 4\sqrt{3}S_m + 6 \cdot \sum m_a^2 - 6 \cdot \sum m_a m_b$
$\iff 6 \cdot \sum m_a m_b \leq 4\sqrt{3}S_m + 5 \cdot \sum m_a^2$
Note the identities $S_m = \frac{3}{4} \cdot S$ and $\sum m_a^2 = \frac{3}{4} \cdot \sum a^2$.
$\iff 6 \cdot \sum m_a m_b \leq 4\sqrt{3}\left(\frac{3}{4} \cdot S\right) + 5 \cdot \left(\frac{3}{4} \cdot \sum a^2\right) \iff 8 \cdot \sum m_a m_b \leq 4\sqrt{3}S + 5 \cdot \sum a^2$
$\iff \frac{2}{3} \cdot \sum m_a \leq \frac{1}{3} \cdot \sqrt{8 \cdot \sum a^2 + 4\sqrt{3}S}$. Note that $\sum GA = \frac{2}{3} \cdot \sum m_a$. $\square$

92. **Author: creatorvn**

The inequality is equivalent to $\frac{a^2 + b^2 - c^2 + R^2}{2ab} \geq 0 \iff \cos C + \frac{R^2}{2ab} \geq 0$
If $\cos C > 0$ the problem has been solved. If not, then the ineq is equivalent to
$\frac{R^2}{2ab} \geq -\cos C = \cos(A + B) \iff 2ab\cos(A + B) \leq R^2$
$\sin A \sin B \sin\left(\frac{\pi}{2} - A - B\right) \leq \frac{1}{8}$, which is true because
$LHS \leq \left(\frac{\sin A + \sin B + \sin\left(\frac{\pi}{2} - A - B\right)}{3}\right)^3 \leq \sin\left(\frac{A + B + \frac{\pi}{2} - A - B}{3}\right)^3 = \frac{1}{8}$. $\square$

92. **Author: Virgil Nicula**

Let the reflection P of A w.r.t. the midpoint M of $[BC]$, i.e. $ABPC$ is a parallelogram $\implies$
$4\left(OB^2 - MB^2\right) = 4 \cdot OM^2 =$
$2\left(OA^2 + OP^2\right) - AP^2 \implies 4R^2 - a^2 = 2\left(R^2 + OP^2\right) - 4m_a^2$
$\implies 2R^2 = a^2 + 2 \cdot OP^2 - 2\left(b^2 + c^2\right) + a^2 \implies OP^2 = b^2 + c^2 + R^2 - a^2 \implies b^2 + c^2 + R^2 \geq a^2$,
with equality iff M is the midpoint of $[AO] \iff b = c = \frac{a}{\sqrt{3}}$. $\square$

92. **Author: Virgil Nicula**

$b^2 + c^2 + R^2 - a^2 \geq 0 \iff 2bc \cdot \cos A + R^2 \geq 0 \iff 8\sin B \sin C \cos A + 1 \geq 0 \iff$
$4\cos A\left[\cos(B - C) + \cos A\right] + 1 \geq 0 \iff 4\cos^2 A + 4\cos(B - C)\cos A + 1 \geq 0 \iff$
$\left[2\cos A + \cos(B - C)\right]^2 + \sin^2(B - C) \geq 0$. Equality holds iff $B = C = 30^\circ$ and $A = 120^\circ$.$\square$

93. **Author: Mateescu Constantin**

We will rewrite the whole inequality in terms of R, r, s by using the identities: $ab+bc+ca = s^2+r^2+4Rr$ and $abc = 4Rrs$
$\left(s^2 + r^2 + 4Rr\right)\left(s^2 + r^2\right) \ge 16Rrs^2 + 36R^2r^2$
$\iff s^4 + s^2\left(2r^2 - 12Rr\right) \ge 36R^2r^2 - 4Rr^3 - r^4$
$\iff \left(s^2 - 6Rr + r^2\right)^2 \ge \left(6Rr - r^2\right)^2 + 36R^2r^2 - 4Rr^3 - r^4$
$\iff \left(s^2 - 6Rr + r^2\right)^2 \ge 72R^2r^2 - 16Rr^3.$
By Gerretsen's Inequality i.e. $s^2 \ge 16Rr - 5r^2$, one gets: $\left(s^2 - 6Rr + r^2\right)^2 \ge \left(10Rr - 4r^2\right)^2$,
Thus it suffices to prove the following inequality $\left(10Rr - 4r^2\right)^2 \ge 72R^2r^2 - 16Rr^3$ which reduces to the obvious one: $(R - 2r)(7R - 2r) \ge 0$. Equality holds iff $\triangle ABC$ is equilateral. □

94. **Author: creatorvn**

$\dfrac{\sum a^2}{\sum ab} - 1 \le \sqrt{1 - \dfrac{2r}{R}} \iff \left(\dfrac{s^2 - 3r^2 - 12Rr}{s^2 + r^2 + 4Rr}\right)^2 \le 1 - \dfrac{2r}{R}$
$LHS \le \left(\dfrac{4R^2 + 3r^2 + 4Rr - 3r^2 - 12Rr}{16Rr - 5r^2 + r^2 + 4Rr}\right)^2 = \left(\dfrac{R^2 - 2Rr}{5Rr - r^2}\right)^2 = \left(\dfrac{1 - 2t}{5t - t^2}\right)^2$ where $t = \dfrac{r}{R}$
We need to prove $\left(\dfrac{1 - 2t}{5t - t^2}\right)^2 \le (1 - 2t)$, which is true, since Euler's Inequality states $t \le \dfrac{1}{2}$. □

95. **Author: creatorvn**

$\dfrac{\frac{2\Delta}{ac}}{\frac{(s-b)(s-a)}{ab}} + \dfrac{\frac{2\Delta}{ab}}{\frac{(s-c)(s-a)}{ac}} \ge 4\dfrac{\sqrt{\frac{s(s-a)}{bc}}}{1 - \sqrt{\frac{(s-b)(s-c)}{bc}}} \iff \dfrac{2\Delta}{s-a}\left(\dfrac{b}{c(s-b)} + \dfrac{c}{b(s-c)}\right) \ge 4\dfrac{\sqrt{s(s-a)}}{\sqrt{bc} - \sqrt{(s-b)(s-c)}}$
$\iff \dfrac{\sqrt{(s-b)(s-c)}}{s-a}\left(\dfrac{b}{c(s-b)} + \dfrac{c}{b(s-c)}\right) \ge 2\dfrac{\sqrt{bc} + \sqrt{(s-b)(s-c)}}{s(s-a)}$
$\iff s\sqrt{(s-b)(s-c)}\left(\dfrac{b}{c(s-b)} + \dfrac{c}{b(s-c)}\right) \ge 2\left(\sqrt{bc} + \sqrt{(s-b)(s-c)}\right)$
By AM-GM, $s \ge \sqrt{bc} + \sqrt{(s-b)(s-c)}$ and $\sqrt{(s-b)(s-c)}\left(\dfrac{b}{c(s-b)} + \dfrac{c}{b(s-c)}\right) \ge 2$
Multiplying them yields the necessary result. □

96. **Author: luisgeometra**

Let $XB = XC = L$. By Ptolemy's theorem for the cyclic quadrilateral $ABXC$, we get
$AB \cdot L + AC \cdot L = AX \cdot BC \implies AX = \dfrac{L(AB + AC)}{BC}$.
By triangle inequality we obtain $XB + XC > BC \implies 2L > BC$
Thus, $AX > \frac{1}{2}(AB + AC)$. Adding the cyclic expressions together yields the result. □

97. **Author: Mateescu Constantin**

Since the points M, I, N are collinear, we will have to find a relationship between the ratios $\dfrac{BM}{MA}$ and $\dfrac{CN}{AN}$. In order to do this, we will express the vectors $\overrightarrow{IM}$ and $\overrightarrow{IN}$ in terms of $\overrightarrow{AB}$ and $\overrightarrow{BC}$ and the collinearity of the former vectors will yield a relationship between the previous ratios.

For convenience, let us denote $\begin{cases} \dfrac{BM}{AM} = k \\ \\ \dfrac{CN}{AN} = q \end{cases}$, where $k, q \in \mathbb{R}_+$. Note: $\begin{cases} \overrightarrow{IA} = \dfrac{b\overrightarrow{BA} + c\overrightarrow{CA}}{2s} \\ \\ \overrightarrow{IB} = \dfrac{c\overrightarrow{CB} + a\overrightarrow{AB}}{2s} \\ \\ \overrightarrow{IC} = \dfrac{a\overrightarrow{AC} + b\overrightarrow{BC}}{2s} \end{cases}$ and:

$$\overrightarrow{IM} = \frac{\overrightarrow{IB} + k \cdot \overrightarrow{IA}}{1+k} = \frac{\left(c\overrightarrow{CB} + a\overrightarrow{AB}\right) + k\left[b\overrightarrow{BA} + c\left(\overrightarrow{CB} + \overrightarrow{BA}\right)\right]}{2s(1+k)}$$
$$= \frac{(a - kb - kc)\overrightarrow{AB} + (-c - kc)\overrightarrow{BC}}{2s(1+k)}$$
$$\overrightarrow{IN} = \frac{\overrightarrow{IC} + q \cdot \overrightarrow{IA}}{1+q} = \frac{\left[a\left(\overrightarrow{AB} + \overrightarrow{BC}\right) + b\overrightarrow{BC}\right] + q\left[b\overrightarrow{BA} + c\left(\overrightarrow{CB} + \overrightarrow{BA}\right)\right]}{2s(1+q)}$$
$$= \frac{(a - qb - qc)\overrightarrow{AB} + (a + b - qc)\overrightarrow{BC}}{2s(1+q)}$$

Therefore, the colinearity of vectors $\overrightarrow{IM}$ and $\overrightarrow{IN}$ implies: $(a - kb - kc)(a + b - qc) = (-c - kc)(a - qb - qc)$ which after expanding is equivalent to $q = \dfrac{a - bk}{c}$. The inequality becomes: $\dfrac{a^2}{4bc} \geq k \cdot \dfrac{a - bk}{c}$
$\iff a^2 \geq 4bk(a - bk) \iff a^2 + 4k^2b^2 \geq 4abk \iff (a - 2kb)^2 \geq 0$, which is clearly true.
Equality is attained iff $a = 2k \cdot b$ i.e. $\dfrac{MB}{AM} = \dfrac{a}{2b}$ and $\dfrac{NC}{AN} = \dfrac{a}{2c}$. $\square$

97. **Author: Virgil Nicula**

Lemma. Let d be a line, three points $\{A, B, C\} \subset d$ and a point $P \notin d$. For another line δ denote intersections K, L, M of δ with the lines PA, PB, PC respectively. Prove that there is the relation $\dfrac{\overline{LA}}{\overline{LP}} \cdot \overline{BC} + \dfrac{\overline{MB}}{\overline{MP}} \cdot \overline{CA} + \dfrac{\overline{NC}}{\overline{NP}} \cdot \overline{AB} = 0$.

Proof. Let d' for which $P \in d'$, $d' \parallel d$. Denote $X \in d \cap \delta$, $Y \in d' \cap \delta$. Thus, $\dfrac{\overline{LA}}{\overline{LP}} \cdot \overline{BC} + \dfrac{\overline{MB}}{\overline{MP}} \cdot \overline{CA} + \dfrac{\overline{NC}}{\overline{NP}} \cdot \overline{AB} = 0 \iff \dfrac{\overline{AX}}{\overline{PY}} \cdot \overline{BC} + \dfrac{\overline{BX}}{\overline{PY}} \cdot \overline{CA} + \dfrac{\overline{CX}}{\overline{PY}} \cdot \overline{AB} = 0 \iff \overline{AX} \cdot \overline{BC} + \overline{BX} \cdot \overline{CA} + \overline{CX} \cdot \overline{AB} = 0$. $\square$

Denote $D \in AI \cap BC$ and apply the lemma. Obtain that $\dfrac{MB}{MA} \cdot DC + \dfrac{NC}{NA} \cdot BD = \dfrac{ID}{IA} \cdot BC \iff b \cdot \dfrac{MB}{MA} + c \cdot \dfrac{NC}{NA} = a$.
In conclusion, $a^2 = \left(b \cdot \dfrac{MB}{MA} + c \cdot \dfrac{NC}{NA}\right)^2 \geq 4 \cdot \left(b \cdot \dfrac{MB}{MA}\right) \cdot \left(c \cdot \dfrac{NC}{NA}\right) = 4bc \cdot \dfrac{MB}{MA} \cdot \dfrac{NC}{NA} \implies \dfrac{MB}{MA} \cdot \dfrac{NC}{NA} \leq \dfrac{a^2}{4bc}$. $\square$

98. **Author: BigSams**

Applying the Hadwiger-Finsler Inequality to $\triangle_m$, $\sum m_a^2 \geq \sum (m_a - m_b)^2 + 4\sqrt{3}S_m$
$\iff 2 \cdot \sum m_a m_b \geq \sum m_a^2 + 4\sqrt{3}S_m$

Note the identities $S_m = \dfrac{3}{4} \cdot S$ and $\sum m_a^2 = \dfrac{3}{4} \cdot \sum a^2$.

$\iff 2 \cdot \sum m_a m_b \geq \dfrac{3}{4} \cdot \sum a^2 + 3\sqrt{3}S \iff \dfrac{4}{3} \cdot \sum m_a m_b \geq \dfrac{1}{2} \cdot \sum a^2 + 2\sqrt{3}S$

$\iff \dfrac{2}{3} \cdot \sum m_a \geq \sqrt{\dfrac{2\left(a^2+b^2+c^2\right) + 4\sqrt{3}S}{3}}$. Note that $\sum GA = \dfrac{2}{3} \cdot \sum m_a$. $\square$

99. **Author: Virgil Nicula**

Denote the midpoint M of $[BC]$ and $N \in AS \cap BC$. Is well-known that $\dfrac{NB}{c^2} = \dfrac{NC}{b^2} = \dfrac{a}{b^2+c^2}$.
Apply van Aubel's relation to S $\dfrac{AS}{b^2+c^2} = \dfrac{SN}{a^2} = \dfrac{AN}{a^2+b^2+c^2}$.
Denote $AM = m_a$, $AN = s_a$, $m\left(\widehat{BAN}\right) = m(\left(\widehat{CAM}\right) = \phi$.

Apply the Sine Law to : $\left\{\begin{array}{ll} \triangle MAC & \dfrac{MC}{\sin\phi} = \dfrac{m_a}{\sin C} \\ \\ \triangle NAB & \dfrac{s_a}{\sin B} = \dfrac{NB}{\sin\phi} \end{array}\right\| \implies \dfrac{s_a}{m_a} = \dfrac{2bc}{b^2+c^2}$

$\implies \dfrac{AS}{AG} = \dfrac{\frac{s_a\left(b^2+c^2\right)}{a^2+b^2+c^2}}{\frac{2m_a}{3}} = \dfrac{3\left(b^2+c^2\right)}{2\left(a^2+b^2+c^2\right)} \cdot \dfrac{s_a}{m_a} = \dfrac{3\left(b^2+c^2\right)}{2\left(a^2+b^2+c^2\right)} \cdot \dfrac{2bc}{b^2+c^2} = \dfrac{3bc}{a^2+b^2+c^2}$ a.s.o.

$\implies \sum \dfrac{AS}{AG} = \dfrac{3(ab+bc+ca)}{a^2+b^2+c^2} \leq 3$. $\square$

100. **Author: Mateescu Constantin**

Note that the inequality can be written as: $a^2 + m^2b^2 \geq m\cos\phi \cdot \left(a^2+b^2-c^2\right) + 4m\sin\phi \cdot \Delta$.

And since $\begin{cases} a^2+b^2-c^2 = 2ab\cos C \\ \\ 2\Delta = ab\sin C \end{cases}$

Our inequality becomes: $a^2 + m^2b^2 \geq 2ab\cos C \cdot m\cos\phi + 2ab\sin C \cdot m\sin\phi$
$\iff a^2 + m^2b^2 \geq 2abm \cdot (\cos C\cos\phi + \sin C\sin\phi) \iff a^2 + m^2b^2 \geq 2ab \cdot m\cos(C-\phi)$,
which is obviously true because $a^2 + m^2b^2 \geq 2abm \geq 2abm\cos(C-\phi)$.
Equality occurs if and only if $a = m \cdot b$ and $\phi = C$. $\square$