Mathematical Excalibur

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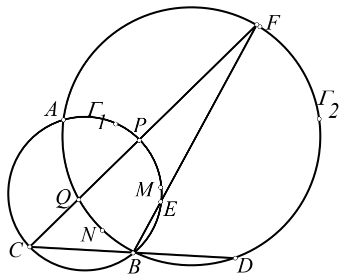
Olympiad Corner

The 2010 Chinese Mathematical Olympiad was held on January. Here are the problems.

Problem 1. As in the figure, two circles Γ_1, Γ_2 intersect at points A, B. A line through B intersects Γ_1, Γ_2 at C, D respectively. Another line through B intersects Γ_1, Γ_2 at E, F respectively. Line CF intersects Γ_1, Γ_2 at P, Q respectively. Let M, N be the midpoints of arcs PB, arc QB respectively. Prove that if $CD = EF$, then C, F, M, N are concyclic.



Problem 2. Let $k \geq 3$ be an integer. Sequence $\{a_n\}$ satisfies $a_k=2k$ and for all $n > k$, $a_n = a_{n-1} + 1$ if a_{n-1} and n are coprime and $a_n=2n$ if a_{n-1} and n are not coprime. Prove that the sequence $\{a_n - a_{n-1}\}$ contains infinitely many prime numbers.

(continued on page 4)

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***April 17, 2010***.

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A Refinement of Bertrand's Postulate

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In this article, we give an elementary demonstration of the famous Bertrand's postulate by using a theorem proved by the mathematician M. El Bachraoni in 2006.

Interesting is the distribution of prime numbers among the natural numbers and problems about their distributions have been stated in very simple ways, but they all turned out to be very difficult. The following *open problem* was stated by the Polish mathematician W. Sierpiński in 1958:

For all natural numbers $n > 1$ and $k \leq n$, there is at least one prime in the range $[kn,(k+1)n]$.

The case k=1 (known as Bertrand's postulate) was stated in 1845 by the French mathematician J. Bertrand and was proved by the Russian mathematician P. L. Chebysev. Simple proofs have been given by the Hungarian mathematician P. Erdos in 1932 and recently by the Romanian mathematician M. Tena [3]. The case k=2 was proved in 2006 by M. El Bachraoni (see [1]). His proof was relatively short and not too complicated. It is freely available on the internet [4].

Below we will present a refinement of Bertrand's postulate and it is perhaps the simplest demonstration of the postulate based on the following

Theorem 1. For any positive integer $n > 1$, there is a prime number between $2n$ and $3n$. (For the proof, see [1] or [4].)

The demonstration in [1] was typical of many theorems in number theory and was based on multiple inequalities valid for large values of n which can be calculated effectively. For the rest of the values of n, there are many basic improvisations, some perhaps difficult to follow.

Theorem 2. For $n \geq 1$, there is a prime number p such that $n < p < 3(n+1)/2$. (Since $3(n+1)/2<2n$ for $n > 3$, this is a refinement of the Bertrand's postulate.)

For the proof, the case n=1 follows from $1<p=2<3$. The case n=2 follows from $2<p=3<9/2$. For n even, say $n=2k$, by Theorem 1, we have a prime p such that $n=2k<p<3k<3(2k+1)/2=3(n+1)/2$. Similarly, for n odd, say $n=2k+1$, we have a prime p such that $n = 2k+1 < 2k+2=2(k+1) < p < 3(k+1)=3(n+1)/2$.

Concerning the distribution of prime numbers among the natural numbers, recently (in 2008) Rafael Jakimczuk has proved a formula (see [2] or [4]) for the n-th prime p_n, which provided a better error term than previous known approximate formulas for p_n. His formula is for $n \geq 4$,

$$p_n = n\log n + \log(n\log n)(n - Li(n\log n)) + \sum_{k=2}^{\infty} \frac{(-1)^k Q_{k-1}(\log(n\log n))}{k!\,n^{k-1}\log^{k-1} n}(n - Li(n\log n))^k + O(h(n)),$$ where

$$Li(x) = \int_2^x \frac{dt}{\log t},\quad h(n) = \frac{n\log^2 n}{\exp(d\sqrt{\log n})}$$

and $Q_{k-1}(x)$ are polynomials.

References

[1] M. El Bachraoni, "*Primes in the Interval [2n,3n]*," Int. J. Contemp. Math. Sciences, vol. 1, 2006, no. 13, 617-621.

[2] R. Jakimczuk, "*An Approximate Formula for Prime Numbers*," Int. J. Contemp. Math. Sciences, vol. 3, 2008, no. 22, 1069-1086.

[3] M. Tena, "*O demonstrație a postulatului lui Bertrand*," G. M.-B 10, 2008.

[4] http://www.m-hikari.com/ijcms.html

Max-Min Inequalities

Pedro Henrique O. Pantoja

(UFRN, NATAL, BRAZIL)

There are many inequalities. In this article, we would like to introduce the readers to some inequalities that involve maximum and minimum.

The first example was a problem from the Federation of Bosnia for Grade 1 in 2008.

Example 1 (*Bosnia-08*) For arbitrary real numbers x, y and z, prove the following inequality:

$$x^2+y^2+z^2-xy-yz-zx \geq \max\left\{\frac{3(x-y)^2}{4},\frac{3(y-z)^2}{4},\frac{3(z-x)^2}{4}\right\}.$$

Solution. Without loss of generality, suppose $x \geq y \geq z$. Then

$$\max\left\{\frac{3(x-y)^2}{4},\frac{3(y-z)^2}{4},\frac{3(z-x)^2}{4}\right\}=\frac{3}{4}(z-x)^2.$$

Let $a = x-y$, $b = y-z$ and $c = z-x$. Then $c = -(a+b)$. Hence, $(z-x)^2 = c^2 = (a+b)^2 = a^2+2ab+b^2$ and

$$\begin{aligned}x^2+y^2+z^2-xy-yz-zx &= \frac{1}{2}[(x-y)^2+(y-z)^2+(z-x)^2]\\ &= \frac{1}{2}(a^2+b^2+a^2+2ab+b^2)\\ &= a^2+ab+b^2.\end{aligned}$$

So it suffices to show

$$a^2+ab+b^2 \geq \frac{3}{4}(a^2+2ab+b^2),$$

which is equivalent to $(a-b)^2 \geq 0$.

The next example was a problem on the 1998 Iranian Mathematical Olympiad.

Example 2. (Iran-98) Let a, b, c, d be positive real numbers such that $abcd$=1. Prove that

$$a^3+b^3+c^3+d^3 \geq \max\left\{a+b+c+d,\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right\}.$$

Solution. It suffices to show

$$a^3+b^3+c^3+d^3 \geq \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}$$

and

$$a^3+b^3+c^3+d^3 \geq a+b+c+d.$$

For the first inequality, we observe that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}=\frac{bcd+acd+abd+abc}{abcd} = bcd+acd+abd+abc.$$

Now, by the AM-GM inequality, we have $a^3+b^3+c^3 \geq 3abc$, $a^3+b^3+d^3 \geq 3abd$, $a^3+c^3+d^3 \geq 3acd$ and $b^3+c^3+d^3 \geq 3bcd$. Adding these four inequalities, we get the first inequality.

Next, let S=a+b+c+d. Then we have

$$S=a+b+c+d \geq 4(abcd)^{1/4}=4$$

by the AM-GM inequality and so $S^3 = S^2S \geq 16S$. The second inequality follows by applying the power mean inequality to obtain

$$\frac{a^3+b^3+c^3+d^3}{4} \geq \left(\frac{a+b+c+d}{4}\right)^3=\frac{S^3}{64} \geq \frac{S}{4}.$$

Example 3. Let a, b, c be positive real numbers. Prove that if $x = \max\{a,b,c\}$ and $y = \min\{a,b,c\}$, then

$$\frac{x}{y}+\frac{y}{x} \geq \frac{18abc}{(a+b+c)(a^2+b^2+c^2)}.$$

Solution. Suppose $a \geq b \geq c$. Then $x = a$ and $y = c$. Using the AM-GM inequality and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\frac{a}{c}+\frac{c}{a} &= \frac{a^2+c^2}{ac}=\frac{(a^2+c^2)b}{abc}\\ &\geq \frac{(2ac)b}{[(a+b+c)/3]^3}=\frac{54abc}{(a+b+c)^3}\\ &\geq \frac{54abc}{3(a^2+b^2+c^2)(a+b+c)}.\end{aligned}$$

The next example was problem 4 in the 2009 USA Mathematical Olympiad.

Example 4. (*USAMO-09*) For $n \geq 2$, let a_1, a_2, …, a_n be positive real numbers such that

$$(a_1+a_2+\cdots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right) \leq \left(n+\frac{1}{2}\right)^2.$$

Prove that

$$\max\{a_1,a_2,\ldots,a_n\} \leq 4\min\{a_1, a_2,\ldots, a_n\}.$$

Solution. Without loss of generality, we may assume

$$m=a_1 \leq a_2 \leq \cdots \leq a_n = M.$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned}\left(n+\frac{1}{2}\right)^2 &\geq (a_1+a_2+\cdots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right)\\ &= (m+a_2+\cdots+M)\left(\frac{1}{M}+\frac{1}{a_2}+\cdots+\frac{1}{m}\right)\\ &\geq \left(\sqrt{\frac{m}{M}}+n-2+\sqrt{\frac{M}{m}}\right)^2.\end{aligned}$$

Taking square root of both sides,

$$n+\frac{1}{2} \geq \sqrt{\frac{m}{M}}+n-2+\sqrt{\frac{M}{m}}.$$

Simplifying, we get $2(m+M) \leq 5\sqrt{mM}$. Squaring both sides, we can get

$$4M^2-17mM+4m^2 \geq 0.$$

Factoring, we see

$$(4M-m)(M-4m) \geq 0.$$

Since $4M-m \geq 0$, we get $M-4m \geq 0$, which is the desired inequality.

The next example was problem 1 on the 2008 Greek National Math Olympiad.

Example 5. (*Greece-08*) For positive integers a_1, a_2, …, a_n, prove that if k=$\max\{a_1,a_2,\ldots,a_n\}$ and t=$\min\{a_1,a_2,\ldots, a_n\}$, then

$$\left(\frac{\sum_{i=1}^n a_i^2}{\sum_{i=1}^n a_i}\right)^{\frac{kn}{t}} \geq \prod_{i=1}^n a_i,$$

When does equality hold?

Solution. By the Cauchy-Schwarz inequality,

$$\left(\sum_{i=1}^n a_i\right)^2 \leq \sum_{i=1}^n 1^2\sum_{i=1}^n a_i^2 = n\sum_{i=1}^n a_i^2.$$

Hence,

$$\frac{\sum_{i=1}^n a_i^2}{\sum_{i=1}^n a_i} \geq \frac{\sum_{i=1}^n a_i}{n}.$$

Since each $a_i \geq 1$, the right side of the above inequality is at least one. Also, we have $kn/t \geq n$. So, applying the above inequality and the AM-GM inequality we have

$$\left(\frac{\sum_{i=1}^n a_i^2}{\sum_{i=1}^n a_i}\right)^{\frac{kn}{t}} \geq \left(\frac{\sum_{i=1}^n a_i}{n}\right)^n \geq \prod_{i=1}^n a_i.$$

Equality holds if and only if all a_i's are equal.

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Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong*. The deadline for sending solutions is ***April 17, 2010.***

Problem 336. (*Due to Ozgur Kircak, Yahya Kemal College, Skopje, Macedonia*) Find all distinct pairs (x,y) of integers satisfying the equation

$$x^3+2009y=y^3+2009x.$$

Problem 337. In triangle ABC, $\angle ABC=\angle ACB=40°$. P and Q are two points inside the triangle such that $\angle PAB=\angle QAC=20°$ and $\angle PCB=\angle QCA=10°$. Determine whether B, P, Q are collinear or not.

Problem 338. Sequences $\{a_n\}$ and $\{b_n\}$ satisfies $a_0=1$, $b_0=0$ and for $n=0,1,2,\ldots$,

$$a_{n+1}=7a_n+6b_n-3,$$
$$b_{n+1}=8a_n+7b_n-4.$$

Prove that a_n is a perfect square for all $n=0,1,2,\ldots$

Problem 339. In triangle ABC, $\angle ACB=90°$. For every n points inside the triangle, prove that there exists a labeling of these points as $P_1, P_2, \ldots, P_n$ such that

$$P_1P_2^2+P_2P_3^2+\cdots+P_{n-1}P_n^2\le AB^2.$$

Problem 340. Let k be a given positive integer. Find the least positive integer N such that there exists a set of $2k+1$ distinct positive integers, the sum of all its elements is greater than N and the sum of any k elements is at most $N/2$.

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Solutions

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Problem 331. For every positive integer n, prove that

$$\sum_{k=0}^{n-1}(-1)^k\cos^n(k\pi/n)=\frac{n}{2^{n-1}}.$$

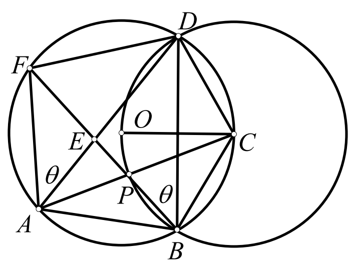
Solution. **Federico BUONERBA** (Università di Roma "Tor Vergata", Roma, Italy), **CHUNG Ping Ngai** (La Salle College, Form 6), **Ovidiu FURDUI** (Campia Turzii, Cluj, Romania), **HUNG Ka Kin Kenneth** (Diocesan Boys' School), **LKL Problem Solving Group** (Madam Lau Kam Lung Secondary School of MFBM), **Paolo PERFETTI** (Math Dept, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Roma, Italy).

Let $\omega=\cos(\pi/n)+i\sin(\pi/n)$. Then we have $\omega^n=-1$ and $(\omega^k+\omega^{-k})/2=\cos(k\pi/n)$. So

$$\sum_{k=0}^{n-1}(-1)^k\cos^n(k\pi/n)=\sum_{k=0}^{n-1}\omega^{kn}\left(\frac{\omega^k+\omega^{-k}}{2}\right)^n$$
$$=\frac{1}{2^n}\sum_{k=0}^{n-1}\omega^{kn}\sum_{j=0}^{n}\binom{n}{j}\omega^{k(n-2j)}$$
$$=\frac{1}{2^n}\sum_{j=0}^{n}\binom{n}{j}\sum_{k=0}^{n-1}(\omega^{2n-2j})^k$$
$$=\frac{1}{2^n}\left(\binom{n}{0}n+\binom{n}{n}n\right)$$
$$=\frac{n}{2^{n-1}}.$$

Problem 332. Let $ABCD$ be a cyclic quadrilateral with circumcenter O. Let BD bisect OC perpendicularly. On diagonal AC, choose the point P such that $PC=OC$. Let line BP intersect line AD and the circumcircle of $ABCD$ at E and F respectively. Prove that PF is the geometric mean of EF and BF in length.

Solution. **HUNG Ka Kin Kenneth** (Diocesan Boys' School) and **Abby LEE** (SKH Lam Woo Memorial Secondary School).



Since $PC=OC=BC$ and ΔBCP is similar to ΔAFP, we have $PF=AF$.

Next, $CB=CD=CP$ implies P is the incenter of ΔABD. Then BF bisects $\angle ABD$ yielding $\angle FAD=\angle ADF$, call it θ. (Alternatively, we have $\angle FAD=\angle PBD=½\angle PCD$. Then

$$\angle AFD=180°-\angle ACD$$
$$=180°-\angle PCD$$
$$=180°-2\angle PBD$$
$$=180°-2\theta.$$

Hence, $\angle ADF=\theta$.) Also, we see $\angle AFE=\angle BFA$ and $\angle EAF=\theta=\angle ADF=\angle ABF$, which imply ΔAFE is similar to ΔBFA. So $AF/EF=BF/AF$. Then

$$PF=AF=\sqrt{EF\times BF}.$$

Comments: For those who are not aware of the incenter characterization used above, they may see *Math Excalibur*, vol. 11, no. 2 for details.

Other commended solvers: **CHOW Tseung Man** (True Light Girls' College), **CHUNG Ping Ngai** (La Salle College, Form 6), **Nicholas LEUNG** (St. Paul's School, London) and **LKL Problem Solving Group** (Madam Lau Kam Lung Secondary School of MFBM).

Problem 333. Find the largest positive integer n such that there exist n 4-element sets $A_1, A_2, \ldots, A_n$ such that every pair of them has exactly one common element and the union of these n sets has exactly n elements.

Solution. **LKL Problem Solving Group** (Madam Lau Kam Lung Secondary School of MFBM).

Let the n elements be 1 to n. For $i=1$ to n, let s_i denote the number of sets in which i appeared. Then $s_1+s_2+\cdots+s_n=4n$. On average, each i appeared in 4 sets.

Assume there is an element, say 1, appeared in more than 4 sets, say 1 is in $A_1, A_2, \ldots, A_5$. Then other than 1, the remaining $3\times5=15$ elements must all be distinct. Now 1 cannot be in all sets, otherwise there would be $3n+1>n$ elements in the union. So there is a set A_6 not containing 1. Its intersections with each of $A_1, A_2, \ldots, A_5$ must be different, yet A_6 only has 4 elements, contradiction. On the other hand, if there is an element appeared in less than 4 sets, then there would be another element appeared in more than 4 sets, contradiction. Hence, every i appeared in exactly 4 sets.

Suppose 1 appeared in A_1, A_2, A_3, A_4. Then we may assume that $A_1=\{1,2,3,4\}$, $A_2=\{1,5,6,7\}$, $A_3=\{1,8,9,10\}$ and $A_4=\{1,11,12,13\}$. Hence, $n\ge13$. Assume $n\ge14$. Then 14 would be in a set A_5. The other 3 elements of A_5 would come from A_1, A_2, A_3, say. Then A_4 and A_5 would have no common element, contradiction.

Hence, n can only be 13. Indeed, for the $n=13$ case, we can take A_1, A_2, A_3, A_4, as above and $A_5=\{2,5,8,11\}$, $A_6=\{2,6,9,12\}$, $A_7=\{2,7,10,13\}$, $A_8=\{3,5,10,12\}$, $A_9=\{3,6,8,13\}$, $A_{10}=\{3,7,9,11\}$, $A_{11}=\{4,5,9,13\}$, $A_{12}=\{4,6,10,11\}$ and $A_{13}=\{4,7,8,12\}$.

Other commended solvers: **CHUNG**

Ping Ngai (La Salle College, Form 6), **HUNG Ka Kin Kenneth** (Diocesan Boys' School) and **Carlo PAGANO** (Università di Roma "Tor Vergata", Roma, Italy).

Problem 324. (*Due to FEI Zhenpeng, Northeast Yucai School, China*) Let $x,y \in (0,1)$ and x be the number whose n-th digit after the decimal point is the n^n-th digit after the decimal point of y for all $n = 1,2,3,\ldots$. Show that if y is rational, then x is rational.

Solution. **CHUNG Ping Ngai** (La Salle College, Form 6),

Since the decimal representation of y is eventually periodic, let L be the length of the period and let the decimal representation of y start to become periodic at the m-th digit. Let k be the least common multiple of $1,2,\ldots,L$. Let n be any integer at least L and $n^n \geq m$.

By the pigeonhole principle, there exist $i < j$ among $0,1,\ldots,L$ such that $n^i \equiv n^j \pmod{L}$. Then for all positive integer d, we have $n^i \equiv n^{i+d(j-i)} \pmod{L}$. Since k is a multiple of $j-i$ and $n \geq L > i$, so we have $n^n \equiv n^{n+k} \pmod{L}$. Since k is also a multiple of L, we have $(n+k)^{n+k} \equiv n^{n+k} \equiv n^n \pmod{L}$. Then the n-th and $(n+k)$-th digit of x are the same. So x is rational.

Other commended solvers: **HUNG Ka Kin Kenneth** (Diocesan Boys' School) and **Carlo PAGANO** (Università di Roma "Tor Vergata", Roma, Italy).

Problem 335. (*Due to Ozgur KIRCAK, Yahya Kemal College, Skopje, Macedonia*) Find all $a \in \mathbb{R}$ for which the functional equation $f\colon \mathbb{R} \to \mathbb{R}$

$$f(x - f(y)) = a(f(x) - x) - f(y)$$

for all $x, y \in \mathbb{R}$ has a unique solution.

Solution. **LE Trong Cuong** (Lam Son High School, Vietnam)

Let $g(x) = f(x)-x$. Then, in terms of g, the equation becomes

$$g(x-y-g(y)) = ag(x)-x.$$

Assume $f(y)=y+g(y)$ is not constant. Let r, s be distinct elements in the range of $f(y)=y+g(y)$. For every real x,

$$g(x-r) = ag(x)-x = g(x-s).$$

This implies $g(x)$ is periodic with period $T=|r-s|>0$. Then

$$\begin{aligned} ag(x) - x &= g(x-y-g(y)) \\ &= g(x+T-y-g(y)) \\ &= ag(x+T) - (x+T) \\ &= ag(x) - x - T. \end{aligned}$$

This implies $T=0$, contradiction. Thus, f is constant, i.e. there exists a real number c so that for all real y, $f(y)=c$. Then the original equation yields $c=a(c-x)-c$ for all real x, which forces $a=0$ and $c=0$.

Other commended solvers: **LKL Problem Solving Group** (Madam Lau Kam Lung Secondary School of MFBM).

Olympiad Corner

(continued from page 1)

Problem 3. Let a,b,c be complex numbers such that for every complex number z with $|z| \leq 1$, we have $|az^2+bz+c| \leq 1$. Find the maximum of $|bc|$.

Problem 4. Let m,n be integers greater than 1. Let $a_1 < a_2 < \cdots < a_m$ be integers. Prove that there exists a subset T of the set of all integers such that the number of elements of T, denoted by $|T|$, satisfies

$$|T| \leq 1 + \frac{a_m - a_1}{2n+1}$$

and for every $i \in \{1,2,\cdots,m\}$, there exist $t \in T$ and $s \in [-n,n]$ such that $a_i = t+s$.

Problem 5. For $n \geq 3$, we place a number of cards at points $A_1, A_2, \cdots, A_n$ and O. We can perform the following operations:

(1) if the number of cards at some point A_i is not less than 3, then we can remove 3 cards from A_i and transfer 1 card to each of the points A_{i-1}, A_{i+1} and O (here $A_0=A_n$, $A_{n+1}=A_1$); or

(2) if the number of cards at O is not less than n, then we can remove n cards from O and transfer 1 card to each $A_1, A_2, \cdots, A_n$.

Prove that if the sum of all the cards placed at these $n+1$ points is not less than n^2+3n+1, then we can always perform finitely many operations so that the number of cards at each of the points is not less than $n+1$.

Problem 6. Let a_1, a_2, a_3, b_1, b_2, b_3 be distinct positive integers satisfying

$$(n+1)a_1^n + na_2^n + (n-1)a_3^n \mid (n+1)b_1^n + nb_2^n + (n-1)b_3^n$$

for all positive integer n. Prove that there exists a positive integer k such that $b_i = ka_i$ for $i=1,2,3$.

Max-Min Inequalities

(continued from page 2)

The inequality in the next example was very hard. It was proposed by Reid Barton and appeared among the 2003 IMO shortlisted problems.

<u>Example 6</u>. Let n be a positive integer and let $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_n)$ be two sequences of positive real numbers. Let $(z_1, z_2, \ldots, z_{2n})$ be a sequence of positive real numbers such that for all $1 \leq i, j \leq n$, $z_{i+j}^2 \geq x_i y_j$. Let $M = \max\{z_1, z_2, \ldots, z_{2n}\}$. Prove that

$$\left(\frac{M + z_2 + \cdots + z_{2n}}{2n}\right)^2 \geq \left(\frac{x_1 + \cdots + x_n}{n}\right)\left(\frac{y_1 + \cdots + y_n}{n}\right).$$

<u>Solution.</u> (*Due to Reid Barton and Thomas Mildorf*) Let

$$X = \max\{x_1, x_2, \ldots, x_n\}$$

and

$$Y = \min\{x_1, x_2, \ldots, x_n\}.$$

By replacing x_i by $x_i' = x_i/X$, y_i by $y_i' = y_i/Y$ and z_i by $z_i' = z_i/(XY)^{1/2}$, we may assume $X=Y=1$. It suffices to prove

$$M + z_2 + \cdots + z_{2n} \geq x_1 + \cdots + x_n + y_1 + \cdots + y_n. \quad (*)$$

Then

$$\frac{M + z_2 + \cdots + z_{2n}}{2n} \geq \frac{1}{2}\left(\frac{x_1 + \cdots + x_n}{n} + \frac{y_1 + \cdots + y_n}{n}\right),$$

which implies the desired inequality by applying the AM-GM inequality to the right side.

To prove (*), we will *<u>claim</u>* that for any $r \geq 0$, the number of terms greater than r on the left side is at least the number of such terms on the right side. Then the k-th largest term on the left side is greater than the k-th largest term on the right side for each k, proving (*).

For $r \geq 1$, there are no terms greater than 1 on the right side. For $r < 1$, let $A = \{i\colon x_i > r\}$, $B = \{j\colon y_j > r\}$, $A+B = \{i+j\colon i \in A, j \in B\}$ and $C = \{k\colon k > 1, z_k > r\}$. Let $|A|$, $|B|$, $|A+B|$, $|C|$ denote the number of elements in A, B, $A+B$, C respectively.

Since $X=Y=1$, so $|A|$, $|B|$ are at least 1. Now $x_i > r$, $y_j > r$ imply $z_{i+j} > r$. So $A+B$ is a subset of C. If A is consisted of $i_1 < \cdots < i_a$ and B is consisted of $j_1 < \cdots < j_b$, then $A+B$ contains

$$i_1+j_1 < i_1+j_2 < \cdots < i_1+j_b < i_2+j_b < \cdots < i_a+j_b.$$

Hence, $|C| \geq |A+B| \geq |A|+|B|-1 \geq 1$. So $z_k > r$ for some k. Then $M > r$. So the left side of (*) has $|C|+1 \geq |A|+|B|$ terms greater than r, which finishes the proof of the claim.