MathLinks.ro

Inequalities Marathon version 1.00

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Author:
MathLink ers

Editor:
Hassan Al-Sibyani

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Introduction

Hello everyone, this file contain 100 problem in inequalities, generally at Pre-Olympiad level. Those problems has been collected from MathLinks.ro at the topic called "'Inequalities Marathon"'. I decide to make this work to be a good reference for young students who are interested in inequalities. This work provides some of the nicest problems among international Olympiads, as well as some amazing problems by the participants. Some of the problems contains more than one solution from the marathon and even outside the marathon from other sources.

THE FILE DESCRIPTION:
I- The next page contain all the members that participate with URLs to their MathLinks.ro user-name information and also that page contain some of the real names of the participants that like to have their real names on this file.
II- Then there are some pages under the title of "'Important Definitions"', this page contain some important definition that you may see them in the problems so if you don't know one of those definitions you can look at it there.
III- Then the problems section comes each problem with its solution(s).

A- The problem format: "'**Problem No.** *'Author'* (Proposer):"'

B- The solution format: "'*Solution No. (Propser)*:"' (Note that the proposer of the solution is not necessary the one who make this solution)

I would like to all the participants of this marathon for making this work, also I would like to give a special thank to Endrit Fejzullahu and Popa Alexandru for their contributing making this file.

If you have another solution to any of those problem or a suggestion or even if you find any mistake of any type on this file please PM me at my MathLinks.ro user or e-mail me at *hasan4444@gawab.com*.

Editor

Participant sorted alphabetically by thier Mathlink username:
Agr 94 Math
alex2008
apratimdefermat
beautifulliar
bokagadha
Brut3Forc3
can hang2007
dgreenb801
Dimitris X
enndb0x
Evariste-Galois
FantasyLover
geniusbliss
great math
hasan4444
keyree10
Mateescu Constantin
mathinequs
Maths Mechanic
Mircea Lascu
Pain rinnegan
peine
Potla
R.Maths
Rofler
saif
socrates
Virgil Nicula
Zhero

List of names:

Name	**Mathlink User**
Apratim De	apratimdefermat
Aravind Srinivas	Agr 94 Math
Dimitris Charisis	Dimitris X
Endrit Fejzullahu	enndb0x
Hassan Al-Sibyani	hasan4444
Hoang Quoc Viet	great math
Mateescu Constantin	Mateescu Constantin
Mharchi Abdelmalek	Evariste-Galois
Mircea Lascu	Mircea Lascu
Mohamed El-Alami	peine
Raghav Grover	Maths Mechanic
Toang Huc Khein	Pain Rinnegan
Popa Alexandru	alex2008
Redwane Khyaoui	R.Maths
Sayan Mukherjee	Potla
Virgil Nicula	Virgil Nicula
Vo Quoc Ba Can	can hang2007

Important Definitions

Abel Summation Formula
Let $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ be two finite sequances of number. Then

$$a_1b_1+a_2b_2+...+a_nb_n = (a_1-a_2)b_1+(a_2-a_3)(b_1+b_2)+\cdots+(a_{n-1}-a_n)(b_1+b_2+\cdots+b_{n-1})+a_n(b_1+b_2+\cdots+b_n)$$

AM-GM (Arithmetic Mean-Geometric Mean) Inequality
If $a_1, a_2, \cdots, a_n$ are nonnegative real numbers, then

$$\frac{a_1, a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

Cauchy-Schwarz Inequality
For any real numbers $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$

$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geq (a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2$$

with equality if and only if a_i and b_i are proportional, $i = 1, 2, \cdots, n$.

Chebyshev's Inequality
If $(a_1, a_2, \cdots, a_n)$ and $(b_1, b_2, \cdots, b_n)$ are two sequences of real numbers arranged in the same order then :

$$n(a_1b_1 + a_2b_2 + ... + a_nb_n) \geq (a_1 + a_2 + ... + a_n)(b_1 + b_2 + ... + b_n)$$

Cyclic Sum
Let n be a positive integer. Given a function f of n variables, define the cyclic sum of variables $(a_1, a_2, \cdots, a_n)$ as

$$\sum_{cyc} f(a_1, a_2, a_3, \ldots a_n) = f(a_1, a_2, a_3, \ldots a_n)+f(a_2, a_3, a_4, \ldots a_n, a_1)+f(a_3, a_4, \ldots a_n, a_1, a_2)+\ldots+f(a_n, a_1, a_2, \ldots a_{n-1})$$

QM-AM-GM-HM (General Mean) Inequality
If $a_1, a_2, \cdots, a_n$ are nonnegative real numbers, then

$$\sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}} \geq \frac{a_1, a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n} \geq \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

GM-HM (Geometric Mean-Harmonic Mean) Inequality
If $a_1, a_2, \cdots, a_n$ are nonnegative real numbers, then

$$\sqrt[n]{a_1 a_2 \cdots a_n} \geq \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

Hlawka's Inequality
Let z_1, z_2, z_3 be three complex numbers. Then:

$$|z_1 + z_2| + |z_2 + z_3| + |z_3 + z_1| \leq |z_1| + |z_2| + |z_3| + |z_1 + z_2 + z_3|$$

Holder's Inequality
Given mn $(m, n \in \mathbb{N})$ positive real numbers $(x_{ij})(i\overline{1,m}\ ,\ j\overline{1,n})$, then :

$$\prod_{i1}^{m} \left(\sum_{j1}^{m} x_{ij} \right)^{\frac{1}{m}} \geq \sum_{j1}^{n} \left(\prod_{i1}^{m} x_{ij}^{\frac{1}{m}} \right)$$

Jensen's Inequality
Let F be a convex function of one real variable. Let $x_1, \ldots, x_n \in \mathbb{R}$ and let $a_1, \ldots, a_n \geq 0$ satisfy $a_1 + \cdots + a_n = 1$. Then:

$$F(a_1 x_1 + \cdots + a_n x_n) \leq a_1 F(x_1) + \cdots + a_n F(x_n)$$

Karamata's Inequality
Given that $F(x)$ is a convex function in x, and that the sequence $\{x_1, \ldots, x_n\}$ majorizes the sequence $\{y_1, \ldots, y_n\}$, the theorem states that

$$F(x_1) + \cdots + F(x_n) \geq F(y_1) + \cdots + F(y_n)$$

The inequality is reversed if $F(x)$ is concave, since in this case the function $-F(x)$ is convex.

Lagrange's Identity
The identity:

$$\left(\sum_{k=1}^{n} a_k b_k \right)^2 = \left(\sum_{k=1}^{n} a_k^2 \right) \left(\sum_{k=1}^{n} b_k^2 \right) - \sum_{1 \leq k \leq j \leq n} (a_k b_j - a_j b_k)^2$$

applies to any two sets $\{a_1, a_2, \cdots, a_n\}$ and $\{b_1, b_2, \cdots, b_n\}$ of real or complex numbers.

Minkowski's Inequality
For any real number $r \geq 1$ and any positive real numbers $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$. Then

$$((a_1+b_1)^r + (a_2+b_2)^r + \cdots + (a_n+b_n)^r)^{\frac{1}{r}} \leq (a_1^r + a_2^r + \cdots + a_n^r)^{\frac{1}{r}} + (b_1^r + b_2^r + \cdots + b_n^r)^{\frac{1}{r}}$$

Muirhead's Inequality
If a sequence A majorizes a sequence B, then given a set of positive reals $x_1, x_2, \cdots, x_n$

$$\sum_{\text{sym}} {x_1}^{a_1} {x_2}^{a_2} \cdots {x_n}^{a_n} \geq \sum_{\text{sym}} {x_1}^{b_1} {x_2}^{b_2} \cdots {x_n}^{b_n}$$

QM-AM (Quadratic Mean-Arithmetic Mean) Inequality
Also called root-mean-square and state that if $a_1, a_2, \cdots, a_n$ are nonnegative real numbers, then

$$\sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}} \geq \frac{a_1, a_2 + \cdots + a_n}{n}$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.**The Rearrangement Inequality**
Let $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$ be real numbers. For any permutation $(a_1', a_2', \cdots, a_n')$ of $(a_1, a_2, \cdots, a_n)$, we have

$$\begin{aligned} a_1b_1 + a_2b_2 + \cdots + a_nb_n &\leq a_1'b_1 + a_2'b_2 + \cdots + a_n'b_n \\ &\leq a_nb_1 + a_{n-1}b_2 + \cdots + a_1b_n \end{aligned}$$

with equality if and only if $(a_1', a_2', \cdots, a_n')$ is equal to $(a_1, a_2, \cdots, a_n)$ or $(a_n, a_{n-1}, \cdots, a_1)$ respectively.

Schur's Inequality
For all non-negative $a, b, c \in \mathbb{R}$ and $r > 0$:

$$a^r(a-b)(a-c) + b^r(b-a)(b-c) + c^r(c-a)(c-b) \geq 0$$

The four equality cases occur when $a = b = c$ or when two of a, b, c are equal and the third is 0.

Common Cases
The $r = 1$ case yields the well-known inequality:$a^3 + b^3 + c^3 + 3abc \geq a^2b + a^2c + b^2a + b^2c + c^2a + c^2b$
When $r = 2$, an equivalent form is: $a^4 + b^4 + c^4 + abc(a+b+c) \geq a^3b + a^3c + b^3a + b^3c + c^3a + c^3b$

Weighted AM-GM Inequality
For any nonnegative real numbers $a_1, a_2, \cdots, a_n$ if $w_1, w_2, \cdots, w_n$ are nonnegative real numbers (weights) with sum 1, then

$$w_1a_1 + w_2a_2 + \cdots + w_na_n \geq a_1^{w_1} a_2^{w_2} \cdots a_n^{w_n}$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

Problem 1 *'India 2002'* (Hassan Al-Sibyani): For any positive real numbers a, b, c show that the following inequality holds

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{c+a}{c+b} + \frac{a+b}{a+c} + \frac{b+c}{b+a}$$

First Solution (Popa Alexandru): Ok. After not so many computations i got that:

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} - \frac{a+b}{c+a} - \frac{b+c}{a+b} - \frac{c+a}{b+c}$$

$$= \frac{abc}{(a+b)(b+c)(c+a)} \left(\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} - \frac{a}{b} - \frac{b}{c} - \frac{c}{a} \right)$$

$$+ \frac{abc}{(a+b)(b+c)(c+a)} \left(\frac{ab}{c^2} + \frac{bc}{a^2} + \frac{ca}{b^2} - 3 \right)$$

So in order to prove the above inequality we need to prove $\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} \geq \frac{a}{b} + \frac{b}{c} + \frac{c}{a}$ and $\frac{ab}{c^2} + \frac{bc}{a^2} + \frac{ca}{b^2} \geq 3$
The second inequality is obvious by AM-GM , and the for the first we have:

$$\left(\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} \right)^2 \geq 3 \left(\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} \right) \geq \left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \right)^2$$

where i used AM-GM and the inequality $3(x^2 + y^2 + z^2) \geq (x + y + z)^2$ for $x = \frac{a}{b}$, $y = \frac{b}{c}$, $z = \frac{c}{a}$
So the inequality is proved.

Second Solution (Raghav Grover): Substitute $\frac{a}{b} = x, \frac{b}{c} = y, \frac{c}{a} = z$ So $xyz = 1$. The inequality after substitution becomes
$x^2z + y^2x + z^2x + x^2 + y^2 + z^2 \geq x + y + z + 3$
$x^2z + y^2x + z^2x \geq 3$ So now it is left to prove that $x^2 + y^2 + z^2 \geq x + y + z$ which is easy.

Third Solution (Popa Alexandru): Bashing out it gives

$$a^4c^2 + b^4a^2 + c^4b^2 + a^3b^3 + b^3c^3 + a^3c^3 \geq abc(ab^2 + bc^2 + ca^2 + 3abc)$$

which is true because AM-GM gives :

$$a^3b^3 + b^3c^3 + a^3c^3 \geq 3a^2b^2c^2$$

and by Muirhead :

$$a^4c^2 + b^4a^2 + c^4b^2 \geq abc(ab^2 + +bc^2 + ca^2)$$

Fourth Solution (Popa Alexandru): Observe that the inequality is equivalent with:

$$\sum_{cyc} \frac{a^2 + bc}{a(a+b)} \geq 3$$

Now use AM-GM:

$$\sum_{cyc} \frac{a^2+bc}{a(a+b)} \geq 3\sqrt[3]{\frac{\prod(a^2+bc)}{abc\prod(a+b)}}$$

So it remains to prove:

$$\prod(a^2+bc) \geq abc\prod(a+b)$$

Now we prove

$$(a^2+bc)(b^2+ca) \geq ab(c+a)(b+c) \Leftrightarrow a^3+b^3 \geq ab^2+a^2b \Leftrightarrow (a+b)(a-b)^2 \geq 0$$

Multiplying the similars we are done.

Problem 2 *'Maxim Bogdan'* (Popa Alexandru): Let $a, b, c, d > 0$ such that $a \leq b \leq c \leq d$ and $abcd = 1$. Then show that:

$$(a+1)(d+1) \geq 3 + \frac{3}{4d^3}$$

Solution (Mateescu Constantin):From the condition $a \leq b \leq c \leq d$ we get that $a \geq \dfrac{1}{d^3}$.

$\Longrightarrow\ (a+1)(d+1) \geq \left(\dfrac{1}{d^3}+1\right)(d+1)$

Now let's prove that $\left(1+\dfrac{1}{d^3}\right)(d+1) \geq 3 + \dfrac{3}{4d^3}$

This is equivalent with: $(d^3+1)(d+1) \geq 3d^3 + \dfrac{3}{4}$

$\Longleftrightarrow [d(d-1)]^2 - [d(d-1)] + 1 \geq \dfrac{3}{4} \Longleftrightarrow \left[d(d-1) - \frac{1}{2}\right]^2 \geq 0.$

Equality holds for $a = \dfrac{1}{d^3}$ and $d(d-1) - \dfrac{1}{2} = 0 \Longleftrightarrow d = \dfrac{1+\sqrt{3}}{2}$

Problem 3 *'Darij Grinberg'* (Hassan Al-Sibyani): If a, b, c are three positive real numbers, then

$$\frac{a}{(b+c)^2} + \frac{b}{(c+a)^2} + \frac{c}{(a+b)^2} \geq \frac{9}{4(a+b+c)}$$

First Solution (Dimitris Charisis):

$\sum \dfrac{a^2}{ab^2+ac^2+2abc} \geq \dfrac{(a+b+c)^2}{\sum_{sym} a^2b + 6abc}$

So we only have to prove that:

$4(a+b+c)^3 \geq 9\sum_{sym} a^2b + 54abc \Longleftrightarrow 4(a^3+b^3+c^3) + 12\sum_{sym} a^2b + 24abc \geq 9\sum_{sym} a^2b + 54abc \Longleftrightarrow$

$4(a^3+b^3+c^3) + 3\sum_{sym} a^2b \geq 30abc$

But $\sum_{sym} a^2b \geq 6abc$ and $a^3+b^3+c^3 \geq 3abc$

So $4(a^3+b^3+c^3) + 3\sum_{sym} a^2b \geq 30abc$

Second Solution (Popa Alexandru): Use Cauchy-Schwartz and Nesbitt:

$$(a+b+c)\left(\frac{a}{(b+c)^2}+\frac{b}{(c+a)^2}+\frac{c}{(a+b)^2}\right)\geq\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)^2\geq\frac{9}{4}$$

Problem 4 *'United Kingdom 1999'* (Dimitris Charisis): For $a,b,c\geq 0$ and $a+b+c=1$ prove that $7(ab+bc+ca)\leq 2+9abc$

First Solution (Popa Alexandru):
Homogenize to

$$2(a+b+c)^3+9abc\geq 7(ab+bc+ca)(a+b+c)$$

Expanding it becomes :

$$\sum_{sym}a^3+6\sum_{sym}a^2b+21abc\geq 7\sum_{sym}a^2b+21abc$$

So we just need to show:

$$\sum_{sym}a^3\geq\sum_{sym}a^2b$$

which is obvious by
$a^3+a^3+b^3\geq 3a^2b$ and similars.

Second Solution (Popa Alexandru): Schur gives $1+9abc\geq 4(ab+bc+ca)$ and use also $3(ab+bc+ca)\leq (a+b+c)^2=1$ Suming is done .

Problem 5 *'Gheorghe Szollosy, Gazeta Matematica'* (Popa Alexandru): Let $x,y,z\in\mathbb{R}_+$. Prove that:

$$\sqrt{x(y+1)}+\sqrt{y(z+1)}+\sqrt{z(x+1)}\leq\frac{3}{2}\sqrt{(x+1)(y+1)(z+1)}$$

Solution (Endrit Fejzullahu): Dividing with the square root on the RHS we have :

$$\sqrt{\frac{x}{(x+1)(z+1)}}+\sqrt{\frac{y}{(x+1)(y+1)}}+\sqrt{\frac{z}{(y+1)(z+1)}}\leq\frac{3}{2}$$

By AM-GM

$$\sqrt{\frac{x}{(x+1)(z+1)}}\leq\frac{1}{2}\left(\frac{x}{x+1}+\frac{1}{y+1}\right)$$
$$\sqrt{\frac{y}{(x+1)(y+1)}}\leq\frac{1}{2}\left(\frac{y}{y+1}+\frac{1}{x+1}\right)$$
$$\sqrt{\frac{z}{(y+1)(z+1)}}\leq\frac{1}{2}\left(\frac{z}{z+1}+\frac{1}{y+1}\right)$$

Summing we obtain

$$LHS\leq\frac{1}{2}\left(\left(\frac{x}{x+1}+\frac{1}{x+1}\right)+\left(\frac{y}{y+1}+\frac{1}{y+1}\right)+\left(\frac{z}{z+1}+\frac{1}{z+1}\right)\right)=\frac{3}{2}$$

Problem 6 *'Romanian Regional Mathematic Olympiad 2006'* (Endrit Fejzullahu): Let a, b, c be positive numbers , then prove that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq \frac{4a}{2a^2+b^2+c^2}+\frac{4b}{a^2+2b^2+c^2}+\frac{4c}{a^2+b^2+2c^2}$$

First Solution (Mateescu Constantin): By $AM-GM$ we have $2a^2+b^2+c^2 \geq 4a\sqrt{bc}$
$\implies \frac{4a}{2a^2+b^2+c^2} \leq \frac{4a}{4a\sqrt{bc}} = \frac{1}{\sqrt{bc}}$
Addind the similar inequalities $\implies RHS \leq \frac{1}{\sqrt{ab}}+\frac{1}{\sqrt{bc}}+\frac{1}{\sqrt{ca}}$ (1)
Using Cauchy-Schwarz we have $\left(\frac{1}{\sqrt{ab}}+\frac{1}{\sqrt{bc}}+\frac{1}{\sqrt{ca}}\right)^2 \leq \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2$
So $\frac{1}{\sqrt{ab}}+\frac{1}{\sqrt{bc}}+\frac{1}{\sqrt{ca}} \leq \frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ (2)
From $(1),(2)$ we obtain the desired result .

Second Solution (Popa Alexandru): By Cauchy-Schwatz :

$$\frac{4a}{2a^2+b^2+c^2} \leq \frac{a}{a^2+b^2}+\frac{a}{a^2+c^2}$$

Then we have

$$RHS \leq \sum_{cyc} \frac{a+b}{a^2+b^2} \leq \sum_{cyc} \frac{2}{a+b} \leq \sum_{cyc}\left(\frac{1}{2a}+\frac{1}{2b}\right) = LHS$$

Third Solution (Popa Alexandru): Since $2a^2+b^2+c^2 \geq a^2+ab+bc+ca = (a+b)(c+a)$. There we have:

$$LHS \leq \sum_{cyc} \frac{4a}{(a+b)(c+a)} = \frac{8(ab+bc+ca)}{(a+b)(b+c)(c+a)} \leq \frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

The last one is equivalent with $(a+b)(b+c)(c+a) \geq 8abc$.

Problem 7 *'Pham Kim Hung'* (Mateescu Constantin): Let a, b, c, d, e be non-negative real numbers such that $a+b+c+d+e=5$. Prove that:

$$abc+bcd+cde+dea+eab \leq 5$$

Solution (Popa Alexandru): Assume $e \leq \min\{a, b, c, d\}$. Then AM-GM gives :

$$e(c+a)(b+d)+bc(a+d-e) \leq \frac{e(5-e)^2}{4}+\frac{(5-2e)^2}{27} \leq 5$$

the last one being equivalent with:

$$(e-1)^2(e+8) \geq 0$$

Problem 8 *'Popa Alexandru'* (Popa Alexandru): Let a, b, c be real numbers such that $0 \leq a \leq b \leq c$. Prove that:

$$(a+b)(c+a)^2 \geq 6abc$$

First Solution (Popa Alexandru): Let

$$b = xa \ , \ c = yb = xya \Rightarrow x, y \geq 1$$

Then:

$$\frac{(a+b)(a+c)^2}{3} \geq 2abc$$
$$\Leftrightarrow (x+1)(xy+1)^2 \cdot a^3 \geq 6x^2ya^3$$
$$\Leftrightarrow (x+1)(xy+1)^2 \geq 6x^2y$$
$$\Leftrightarrow (x+1)(4xy+(xy-1)^2) \geq 6x^2y$$
$$\Leftrightarrow 4xy + (xy-1)^2 \cdot x + (xy-1)^2 - 2x^2y \geq 0$$

We have that:

$$4xy + (xy-1)^2 \cdot x + (xy-1)^2 - 2x^2y \geq$$
$$\geq 4xy + 2(xy-1)^2 - 2x^2y (\text{ because } x \geq 1)$$
$$= 2x^2y^2 + 2 - 2x^2y = 2xy(y-1) + 2 > 0$$

done.

Second Solution (Endrit Fejzullahu): Let $b = a + x, c = b + y = a + x + y$,sure $x, y \geq 0$
Inequality becomes

$$(2a+x)(x+y+2a)^2 - 6a(a+x)(a+x+y) \geq 0$$

But

$$(2a+x)(x+y+2a)^2 - 6a(a+x)(a+x+y) = 2a^3 + 2a^2y + 2axy + 2ay^2 + x^3 + 2x^2y + xy^2$$

which is clearly positive.

Problem 9 *'Nesbitt'* (Raghav Grover): Prove for positive reals

$$\frac{a}{b+c} + \frac{b}{c+d} + \frac{c}{d+a} + \frac{d}{a+b} \geq 2$$

Solution (Dimitris Charisis):
From andreescu $LHS \geq \dfrac{(a+b+c+d)^2}{\sum\limits_{sym} ab + (ac+bd)}$
So we only need to prove that:

$$(a+b+c+d)^2 \geq 2\sum_{sym} ab + 2(ac+bd) \Longleftrightarrow (a-c)^2 + (b-d)^2 \geq 0....$$

Problem 10 *'Vasile Cartoaje'* (Dimitris Charisis): Let a, b, c, d be REAL numbers such that $a^2+b^2+c^2+d^2 = 4$ Prove that:

$$a^3 + b^3 + c^3 + d^3 \leq 8$$

Solution (Popa Alexandru): Just observe that

$$a^3 + b^3 + c^3 + d^3 \leq 2(a^2 + b^2 + c^2 + d^2) = 8$$

because $a, b, c, d \leq 2$

Problem 11 *'Mircea Lascu'* (Endrit Fejzullahu): Let a, b, c be positive real numbers such that $abc = 1$ .Prove that

$$\sum_{cyc} \frac{1}{a^2 + 2b^2 + 3} \leq \frac{1}{2}$$

Solution (Popa Alexandru): Using AM-GM we have :

$$LHS = \sum_{cyc} \frac{1}{(a^2 + b^2) + (b^2 + 1) + 2} \leq \sum_{cyc} \frac{1}{2ab + 2b + 2}$$

$$= \frac{1}{2} \sum_{cyc} \frac{1}{ab + b + 1} = \frac{1}{2}$$

because

$$\frac{1}{bc + c + 1} = \frac{1}{bc + c + abc} = \frac{1}{c} \cdot \frac{1}{ab + b + 1} = \frac{ab}{ab + b + 1}$$

and

$$\frac{1}{ca + a + 1} = \frac{1}{\frac{1}{b} + a + 1} = \frac{b}{ab + b + 1}$$

so

$$\sum_{cyc} \frac{1}{ab + b + 1} = \frac{1}{ab + b + 1} + \frac{ab}{ab + b + 1} + \frac{b}{ab + b + 1} = 1$$

Problem 12 *'Popa Alexandru'* (Popa Alexandru): Let $a, b, c > 0$ such that $a + b + c = 1$. Prove that:

$$\frac{1 + a + b}{2 + c} + \frac{1 + b + c}{2 + a} + \frac{1 + c + a}{2 + b} \geq \frac{15}{7}$$

First Solution (Dimitris Charisis):

$$\sum \frac{1 + a + b}{2 + c} + 1 \geq \frac{15}{7} + 3 \iff \sum \frac{3 + (a + b + c)}{2 + c} \geq \frac{36}{7} \iff \sum \frac{4}{2 + c} \geq \frac{36}{7}$$

But $\sum \frac{2^2}{2+c} \geq \frac{(2+2+2)^2}{2+2+2+a+b+c} = \frac{36}{7}$

Second Solution (Endrit Fejzullahu): Let $a \geq b \geq c$ then by Chebyshev's inequality we have
$LHS \geq \frac{1}{3}\left(1+1+1+2(a+b+c)\right)\sum_{cyc} \frac{1}{2+a} = \frac{5}{3}\sum_{cyc} \frac{1}{2+a}$
By Titu's Lemma $\sum_{cyc} \frac{1}{2+a} \geq \frac{9}{7}$, then $LHS \geq \frac{15}{7}$

Problem 13 *'Titu Andreescu, IMO 2000'* (Dimitris Charisis): Let a, b, c be positive so that $abc = 1$

$$\left(a-1+\frac{1}{b}\right)\left(b-1+\frac{1}{c}\right)\left(c-1+\frac{1}{a}\right) \leq 1$$

Solution (Endrit Fejzullahu):
$\left(a-1+\frac{1}{b}\right)\left(b-1+\frac{1}{c}\right)\left(c-1+\frac{1}{a}\right) \leq 1$
Substitute $a = \frac{x}{y}, b = \frac{y}{z}$
Inequality is equivalent with
$\left(\frac{x}{y}-1+\frac{z}{y}\right)\left(\frac{y}{z}-1+\frac{x}{z}\right)\left(\frac{z}{x}-1+\frac{y}{x}\right) \leq 1$
$\iff (x+z-y)(y-z+x)(z-x+y) \leq xyz$
WLOG ,Let $x > y > z$, then $x+z > y, x+y > z$.If $y+z < x$, then we are done because
$(x+z-y)(y-z+x)(z-x+y) \leq 0$ and $xyz \geq 0$
Otherwise if $y+z > x$, then x, y, z are side lengths of a triangle ,and then we can make the substitution
$x = m+n, y = n+t$ and $z = t+m$
Inequality is equivalent with
$8mnt \leq (m+n)(n+t)(t+m)$, this is true by AM-GM
$m+n \geq 2\sqrt{mn}, n+t \geq 2\sqrt{nt}$ and $t+m \geq 2\sqrt{tm}$, multiply and we're done.

Problem 14 *'Korea 1998'* (Endrit Fejzullahu): Let $a, b, c > 0$ and $a+b+c = abc$. Prove that:

$$\frac{1}{\sqrt{a^2+1}} + \frac{1}{\sqrt{b^2+1}} + \frac{1}{\sqrt{c^2+1}} \leq \frac{3}{2}$$

First Solution (Dimitris Charisis): Setting $a = \frac{1}{x}, b = \frac{1}{y}, c = \frac{1}{z}$ the condition becomes $xy+yz+zx = 1$, and the inequality:
$\sum \frac{x}{\sqrt{x^2+1}} \leq \frac{3}{2}$
But $\sum \frac{x}{\sqrt{x^2+1}} = \sum \frac{x}{\sqrt{x^2+xy+xz+zy}} = \sum \sqrt{\frac{x}{x+y}\frac{x}{x+z}}$
But $\sqrt{\frac{x}{x+y}\frac{x}{x+z}} \leq \frac{\frac{x}{x+y}+\frac{x}{x+z}}{2}$

So $\sum \sqrt{\frac{x}{x+y}\frac{x}{x+z}} \leq \frac{\frac{x}{x+y}+\frac{y}{x+y}+\frac{x}{z+x}+\frac{z}{z+x}+\frac{y}{y+z}+\frac{z}{z+y}}{2} = \frac{3}{2}$

Second Solution (Raghav Grover):
Substitute $a = \tan x$,$b = \tan y$ and $c = \tan z$ where $x + y + z = \pi$
And we are left to prove
$\cos x + \cos y + \cos z \leq \frac{3}{2}$
Which i think is very well known..

Third Solution (Endrit Fejzullahu): By AM-GM we have $a + b + c \geq 3\sqrt[3]{abc}$ and since $a + b + c = abc \implies (abc)^2 \geq 27$
We rewrite the given inequality as
$\frac{1}{3}\left(\frac{1}{a^2+1}+\frac{1}{b^2+1}+\frac{1}{c^2+1}\right) \leq \frac{1}{2}$
Since function $f(a) = \frac{1}{\sqrt{a^2+1}}$ is concave ,we apply Jensen's inequality
$\frac{1}{3}f(a)+\frac{1}{3}f(b)+\frac{1}{3}f(c) \leq f\left(\frac{a+b+c}{3}\right) = f\left(\frac{abc}{3}\right) = \frac{1}{\sqrt{\frac{(abc)^2}{3^2}+1}} \leq \frac{1}{2} \iff (abc)^2 \geq 27, \mathbb{QED}$

Problem 15 *'—'* (Dimitris Charisis):
If $a, b, c \in \mathbb{R}$ and $a^2 + b^2 + c^2 = 3$. Find the minimum value of $A = ab + bc + ca - 3(a + b + c)$.

Solution (Endrit Fejzullahu):
$ab + bc + ca = \frac{(a+b+c)^2 - a^2 - b^2 - c^2}{2} = \frac{(a+b+c)^2 - 3}{2}$
Let $a + b + c = x$
Then $A = \frac{x^2}{2} - 3x - \frac{3}{2}$
We consider the second degree fuction $f(x) = \frac{x^2}{2} - 3x - \frac{3}{2}$
We obtain minimum for $f\left(\frac{-b}{2a}\right) = f(3) = -6$
Then $A_{min} = -6$,it is attained for $a = b = c = 1$

Problem 16 *'Cezar Lupu'* (Endrit Fejzullahu): If a, b, c are positive real numbers such that $a+b+c = 1$.Prove that

$$\frac{a}{\sqrt{b+c}}+\frac{b}{\sqrt{c+a}}+\frac{c}{\sqrt{a+b}} \geq \sqrt{\frac{3}{2}}$$

First Solution (keyree10): Let $f(x) = \frac{x}{\sqrt{1-x}}$. $f"(x) > 0$
Therefore, $\sum \frac{a}{\sqrt{1-a}} \geq \frac{3s}{\sqrt{1-s}}$,where $s = \frac{a+b+c}{3} = \frac{1}{3}$, by jensen's.
$\implies \frac{a}{\sqrt{b+c}}+\frac{b}{\sqrt{c+a}}+\frac{c}{\sqrt{a+b}} \geq \sqrt{\frac{3}{2}}$. Hence proved

Second Solution (geniusbliss):
By holders' inequality,
$$\left(\sum_{cyclic}\frac{a}{(b+c)^{\frac{1}{2}}}\right)\left(\sum_{cyclic}\frac{a}{(b+c)^{\frac{1}{2}}}\right)\left(\sum_{cyclic}a(b+c)\right)\geq(a+b+c)^3$$
thus,
$$\left(\sum_{cyclic}\frac{a}{(b+c)^{\frac{1}{2}}}\right)^2\geq\frac{(a+b+c)^2}{2(ab+bc+ca)}\geq\frac{3(ab+bc+ca)}{2(ab+bc+ca)}=\frac{3}{2}$$
or,
$$\frac{a}{\sqrt{b+c}}+\frac{b}{\sqrt{c+a}}+\frac{c}{\sqrt{a+b}}\geq\sqrt{\frac{3}{2}}$$
Third Solution (Redwane Khyaoui):
$$\frac{a}{\sqrt{b+c}}+\frac{b}{\sqrt{c+a}}+\frac{c}{\sqrt{a+b}}\geq\frac{1}{3}(a+b+c)\left(\frac{1}{\sqrt{b+c}}+\frac{1}{\sqrt{c+a}}+\frac{1}{\sqrt{a+b}}\right)$$
$f(x)=\frac{1}{\sqrt{x}}$ is a convex function, so jensen's inequality gives:
$$LHS\geq\frac{1}{3}\left(3\cdot\frac{1}{\sqrt{\frac{2}{3}(a+b+c)}}\right)=\sqrt{\frac{3}{2}}$$

Problem 17 *'Adapted after IMO 1987 Shortlist'* (keyree10): If a,b,c are REALS such that $a^2+b^2+c^2=1$
Prove that
$$a+b+c-2abc\leq\sqrt{2}$$

Solution (Popa Alexandru):
Use Cauchy-Schwartz:
$$LHS=a(1-2bc)+(b+c)\leq\sqrt{(a^2+(b+c)^2)((1-2bc)^2+1)}$$
So it'll be enough to prove that :
$$(a^2+(b+c)^2)((1-2bc)^2+1)\leq2\Leftrightarrow(1+2bc)(1-bc+2b^2c^2)\leq1\Leftrightarrow4b^2c^2\leq1$$
which is true because
$$1\geq b^2+c^2\geq2bc$$
done

Problem 18 *'Russia 2000'* (Popa Alexandru): Let $x,y,z>0$ such that $xyz=1$. Show that:
$$x^2+y^2+z^2+x+y+z\geq2(xy+yz+zx)$$

First Solution (Hoang Quoc Viet): To solve the problem of alex, we need Schur and Cauchy inequality as demonstrated as follow

$$a+b+c \geq 3\sqrt[3]{abc} = \sqrt[3]{(abc)^2} \geq \frac{9abc}{a+b+c} \geq 2(ab+bc+ca)-(a^2+b^2+c^2)$$

Note that we possess the another form of Schur such as

$$(a^2+b^2+c^2)(a+b+c)+9abc \geq 2(ab+bc+ca)(a+b+c)$$

Therefore, needless to say, we complete our proof here.

Second Solution (Popa Alexandru): Observe that :

$$2(xy+yz+zx)-(x^2+y^2+z^2) = (\sqrt{x}+\sqrt{y}+\sqrt{z})(\sqrt{x}+\sqrt{y}-\sqrt{z})(\sqrt{x}-\sqrt{y}+\sqrt{z})(-\sqrt{x}+\sqrt{y}+\sqrt{z})$$
$$\leq \sqrt{xyz}(\sqrt{x}+\sqrt{y}+\sqrt{z}) = \sqrt{x}+\sqrt{y}+\sqrt{z} \leq x+y+z$$

so the conclusion follows .

Third Solution (Mohamed El-Alami): Let put $p = x+y+z$ and $q = xy+yz+zx$, so then we can rewrite our inequality as $p^2+p \geq 2q$. Using Schur's inequality we have that $4q \leq \frac{p^3+9}{p}$, so it'll be enough to prove that $p^3+9 \leq p^3+p^2 \Leftrightarrow p^2 \geq 9$, which is true by AM - GM .

Problem 19 *'Hoang Quoc Viet'* (Hoang Quoc Viet): Let a, b, c be positive reals satisfying $a^2+b^2+c^2 = 3$. Prove that

$$(abc)^2(a^3+b^3+c^3) \leq 3$$

First Solution (Hoang Quoc Viet): Let

$$A = (abc)^2(a^3+b^3+c^3)$$

Therefore, we only need to maximize the following expression

$$A^3 = (abc)^6(a^3+b^3+c^3)^3$$

Using Cauchy inequality as follows, we get

$$A^3 = \frac{1}{3^6}\left(3a^2b\right)\left(3a^2c\right)\left(3b^2a\right)\left(3b^2c\right)\left(3c^2a\right)\left(3c^2b\right)\left(a^3+b^3+c^3\right)^3 \leq \left(\frac{3(a^2+b^2+c^2)(a+b+c)}{9}\right)^9$$

It is fairly straightforward that

$$a+b+c \leq \sqrt{3(a^2+b^2+c^2)} = 3$$

Therefore,

$$A^3 \leq 3^3$$

which leads to $A \leq 3$ as desired. The equality case happens $\iff a=b=c=1$

Second Solution (FantasyLover):
For the sake of convenience, let us introduce the new unknowns u, v, w as follows:

$$\begin{aligned} u &= a+b+c \\ v &= ab+bc+ca \\ w &= abc \end{aligned}$$

Now note that $u^2 - 2v = 3$ and $a^3 + b^3 + c^3 = u(u^2 - 3v) = u\left(\frac{9-u^2}{2}\right)$.
We are to prove that $w^2\left(u \cdot \frac{9-u^2}{2} + 3w\right) \leq 3$.
By AM-GM, we have $\sqrt[3]{abc} \leq \frac{a+b+c}{3} \implies w \leq \frac{u^3}{3^3}$.
Hence, it suffices to prove that $u^7 \cdot \frac{9-u^2}{2} + \frac{u^9}{3^2} \leq 3^7$.
However, by QM-AM we have $\sqrt{\frac{a^2+b^2+c^2}{3}} \geq \frac{a+b+c}{3} \implies u \leq 3$
differentiating, u achieves its maximum when $\frac{7u^6(9-u^2)}{2} = 0$.
Since a, b, c are positive, u cannot be 0, and the only possible value for u is 3.
Since $u \leq 3$, the above inequality is true.

Problem 20 *'Murray Klamkin, IMO 1983'* (Hassan Al-Sibyani): Let a, b, c be the lengths of the sides of a triangle. Prove that:

$$a^2b(a-b) + b^2c(b-c) + c^2a(c-a) \geq 0$$

First Solution (Popa Alexandru): Use Ravi substitution $a = x+y$, $b = y+z$, $c = z+x$ then the inequality becomes :

$$\frac{x^2}{y} + \frac{y^2}{z} + \frac{z^2}{x} \geq x+y+z \ ,$$

true by Cauchy-Schwartz.

Second Solution (geniusbliss): we know from triangle inequality that $b \geq (a-c)$ and $c \geq (b-a)$ and $a \geq (c-b)$
therefore,
$a^2b(a-b) + b^2c(b-c) + c^2a(c-a) \geq a^2(a-c)(a-b) + b^2(b-a)(b-c) + c^2(c-b)(c-a) \geq 0$
and the last one is schur's inequality for $r = 2$ so proved with the equality holding when $a = b = c$ or for and equilateral triangle

Third Solution (Popa Alexandru): The inequality is equivalent with :

$$\frac{1}{2}((a-b)^2(a+b-c)(b+c-a) + (b-c)^2(b+c-a)(a+c-b) + (c-a)^2(a+c-b)(a+b-c)) \geq 0$$

Problem 21 *'Popa Alexandru'* (Popa Alexandru): Let $x, y, z \in \left[\frac{1}{3}, \frac{2}{3}\right]$. Show that :

$$1 \geq \sqrt[3]{xyz} + \frac{2}{3(x+y+z)}$$

First Solution (Endrit Fejzullahu):
By Am-Gm $x+y+z \geq 3\sqrt[3]{xyz}$,then
$\frac{2}{3(x+y+z)} + \sqrt[3]{xyz} \leq \frac{2}{9}\sqrt[3]{xyz} + \sqrt[3]{xyz}$
Since $x, y, z \in \left[\frac{1}{3}, \frac{2}{3}\right]$,then $\frac{1}{3} \leq \sqrt[3]{xyz} \leq \frac{2}{3}$
Let $a = \sqrt[3]{xyz}$,then $a + \frac{2}{9a} \leq 1 \iff 9a^2 - 9a + 2 \leq 0 \iff 9\left(a - \frac{1}{3}\right)\left(a - \frac{2}{3}\right) \leq 0$,we're done since $\frac{1}{3} \leq a \leq \frac{2}{3}$

Second Solution (Popa Alexandru): Let $s = a+b+c$. By AM-GM it is enough to prove that :

$$1 - \frac{s}{3} - \frac{2}{3s} \geq 0 \Leftrightarrow (s-1)(2-s) \geq 0$$

The last one is true since $x, y, z \in \left[\frac{1}{3}, \frac{2}{3}\right]$

Problem 22 *'Endrit Fejzullahu'* (Endrit Fejzullahu): Let a, b, c be side lengths of a triangle,and β is the angle between a and c.Prove that

$$\frac{b^2+c^2}{a^2} > \frac{2\sqrt{3}c\sin\beta - a}{b+c}$$

Solution (Endrit Fejzullahu): According to the Weitzenbock's inequality we have
$a^2+b^2+c^2 \geq 4\sqrt{3}S$ and $S = \frac{ac\sin\beta}{2}$
Then
$b^2+c^2 \geq 2\sqrt{3}ac\sin\beta - a^2$,dividing by a^2 , we have
$\frac{b^2+c^2}{a^2} \geq \frac{2\sqrt{3}c\sin\beta - a}{a}$
Since $a < b+c \implies \frac{b^2+c^2}{a^2} \geq \frac{2\sqrt{3}c\sin\beta - a}{a} > \frac{2\sqrt{3}c\sin\beta - a}{b+c}$

Problem 23 *'Dinu Serbanescu, Junior TST 2002, Romania'* (Hassan Al-Sibyani): If $a, b, c \in (0,1)$ Prove that:

$$\sqrt{abc} + \sqrt{(1-a)(1-b)(1-c)} < 1$$

First Solution (FantasyLover):
Since $a, b, c \in (0,1)$, let us have $a = \sin^2 A, b = \sin^2 B, c = \sin^2 C$ where $A, B, C \in \left(0, \frac{\pi}{2}\right)$.
Then, we are to prove that $\sin A \sin B \sin C + \cos A \cos B \cos C < 1$.
Now noting that $\sin C, \cos C < 1$, we have $\sin A \sin B \sin C + \cos A \cos B \cos C < \sin A \sin B + \cos A \cos B = \cos(A - B) \leq 1$. $\blacksquare$

Second Solution (Popa Alexandru): Cauchy-Schwartz and AM-GM works fine :

$$\sqrt{abc} + \sqrt{(1-a)(1-b)(1-c)} = \sqrt{a}\sqrt{bc} + \sqrt{1-a}\sqrt{(1-b)(1-c)} \leq$$
$$\sqrt{a + (1-a)}\sqrt{bc + (1-b)(1-c)} = \sqrt{bc + (1-b)(1-c)} < 1$$

Third Solution (Redwane Khyaoui):

$$\sqrt{abc} + \sqrt{(1-a)(1-b)(1-c)} \leq \frac{a + bc}{2} + \frac{1 - a + (1-b)(1-c)}{2}$$

So we only need to prove that $\dfrac{a + bc}{2} + \dfrac{1 - a + (1-b)(1-c)}{2} \leq 1$
$\Leftrightarrow \dfrac{1}{b} + \dfrac{1}{c} \geq 2$ which is true since $a, b \in [0,1]$

Problem 24 *'Hojoo Lee'* (FantasyLover): For all positive real numbers a, b, c, prove the following:

$$\frac{1}{\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1}} - \frac{1}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}} \geq \frac{1}{3}$$

First Solution (Popa Alexandru): Using p, q, r substitution ($p = a + b + c$, $q = ab + bc + ca$, $r = abc$) the inequality becomes :

$$\frac{3(p + q + r + 1)}{2p + q + r} \geq \frac{9 + 3r}{q} \Leftrightarrow pq + 2q^2 \geq 6pr + 9r$$

which is true because is well-known that $pq \geq 9r$ and $q^2 \geq 3pr$

Second Solution (Endrit Fejzullahu): After expanding the inequality is equivalent with :

$$\frac{1}{a(a+1)} + \frac{1}{b(b+1)} + \frac{1}{c(c+1)} \geq \frac{1}{3}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)\left(\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1}\right)$$

This is true by Chebyshev's inequality , so we're done .

Problem 25 *'Mihai Opincariu'* (Popa Alexandru): Let $a, b, c > 0$ such that $abc = 1$. Prove that :

$$\frac{ab}{a^2 + b^2 + \sqrt{c}} + \frac{bc}{b^2 + c^2 + \sqrt{a}} + \frac{ca}{c^2 + a^2 + \sqrt{b}} \leq 1$$

First Solution (FantasyLover):
We have $a^2+b^2+\sqrt{c}\geq 2ab+\sqrt{c}=\frac{2}{c}+\sqrt{c}$.
Hence, it suffices to prove that $\displaystyle\sum_{\text{cyc}}\frac{\frac{1}{a}}{\frac{2}{a}+\sqrt{a}}=\sum_{\text{cyc}}\frac{1}{2+a\sqrt{a}}\leq 1$.
Reducing to a common denominator, we prove that

$$\sum_{\text{cyc}}\frac{1}{2+a\sqrt{a}}=\frac{4(a\sqrt{a}+b\sqrt{b}+c\sqrt{c})+ab\sqrt{ab}+bc\sqrt{bc}+ca\sqrt{ca}+12}{4(a\sqrt{a}+b\sqrt{b}+c\sqrt{c})+2(ab\sqrt{ab}+bc\sqrt{bc}+ca\sqrt{ca})+9}\leq 1$$

.
Rearranging, it remains to prove that $ab\sqrt{ab}+bc\sqrt{bc}+ca\sqrt{ca}\geq 3$.
Applying AM-GM, we have $ab\sqrt{ab}+bc\sqrt{bc}+ca\sqrt{ca}\geq 3\sqrt[3]{a^2b^2c^2\sqrt{a^2b^2c^2}}=3$, and we are done. ■

Second Solution (Popa Alexandru):

$$LHS\leq\sum_{cyc}\frac{ab}{2ab+\sqrt{c}}=\sum_{cyc}\frac{1}{2+c\sqrt{c}}=\sum_{cyc}\frac{1}{2+\frac{x}{y}}\leq RHS$$

Problem 26 *'Korea 2006 First Examination'* (FantasyLover): x,y,z are real numbers satisfying the condition $3x+2y+z=1$. Find the maximum value of

$$\frac{1}{1+|x|}+\frac{1}{1+|y|}+\frac{1}{1+|z|}$$

.

Solution (dgreenb801):
We can assume x,y, and z are all positive, because if one was negative we could just make it positive, which would allow us to lessen the other two variables, making the whole sum larger.
Let $3x=a$, $2y=b$, $z=c$, then $a+b+c=1$ and we have to maximize
$\frac{3}{a+3}+\frac{2}{b+2}+\frac{1}{c+1}$
Note that
$\left(\frac{3}{a+c+3}+1\right)-\left(\frac{3}{a+3}+\frac{1}{c+1}\right)=\frac{a^2c+ac^2+6ac+6c}{(a+3)(c+1)(a+c+3)}\geq 0$
So for fixed $a+c$, the sum is maximized when $c=0$.
We can apply the same reasoning to show the sum is maximized when $b=0$.
So the maximum occurs when $a=1$, $b=0$, $c=0$, and the sum is $\frac{11}{4}$.

Problem 27 *'Balkan Mathematical Olympiad 2006'* (dgreenb801):

$$\frac{1}{a(1+b)}+\frac{1}{b(1+c)}+\frac{1}{c(1+a)}\geq\frac{3}{1+abc}$$

for all positive reals.

First Solution (Popa Alexandru):
AM-GM works :

$$(1+abc)\,LHS+3=\sum_{cyc}\frac{1+abc+a+ab}{a+ab}=\sum_{cyc}\frac{1+a}{ab+a}+\sum_{cyc}\frac{b(c+1)}{b+1}\geq\frac{3}{\sqrt[3]{abc}}+3\sqrt[3]{abc}\geq 6$$

Second Solution (Popa Alexandru): Expanding it is equivalent with :

$$ab(b+1)(ca-1)^2+bc(c+1)(ab-1)^2+ca(a+1)(bc-1)^2\geq 0$$

Problem 28 *'Junior TST 2007, Romania'* (Popa Alexandru): Let $a,b,c>0$ such that $ab+bc+ca=3$. Show that :

$$\frac{1}{1+a^2(b+c)}+\frac{1}{1+b^2(c+a)}+\frac{1}{1+c^2(a+b)}\leq\frac{1}{abc}$$

First Solution (Endrit Fejzullahu): Since $ab+bc+ca=3\implies abc\leq 1$
Then $1+3a-abc\geq 3a$
Then
$\sum\frac{1}{1+3a-abc}\leq\frac{1}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=\frac{ab+bc+ca}{3abc}=\frac{1}{abc}$
Second Solution (Popa Alexandru): Using $abc\leq 1$ and the condition we have :

$$\sum_{cyc}\frac{1}{1+a^2\left(b+c\right)}\leq\frac{1}{abc}\Leftrightarrow\sum_{cyc}\frac{1}{\frac{1}{abc}+a\left(\frac{1}{b}+\frac{1}{c}\right)}\leq 1\Leftrightarrow\sum_{cyc}\frac{1}{1+a\left(\frac{1}{b}+\frac{1}{c}\right)}\leq 1\Leftrightarrow\sum_{cyc}\frac{1}{1+a\left(\frac{3}{abc}-\frac{1}{a}\right)}\leq 1\Leftrightarrow\sum_{cyc}\frac{ab}{3}\leq 1$$

Problem 29 *'Lithuania 1987'* (Endrit Fejzullahu): Let a,b,c be positive real numbers .Prove that

$$\frac{a^3}{a^2+ab+b^2}+\frac{b^3}{b^2+bc+c^2}+\frac{c^3}{c^2+ca+a^2}\geq\frac{a+b+c}{3}$$

First Solution (dgreenb801): By Cauchy,
$\sum\frac{a^3}{a^2+ab+b^2}=\sum\frac{a^4}{a^3+a^2b+ab^2}\geq\frac{(a^2+b^2+c^2)^2}{a^3+b^3+c^3+a^2b+ab^2+b^2c+bc^2+c^2a+ca^2}$
This is $\geq\frac{a+b+c}{3}$ if
$(a^4+b^4+c^4)+2(a^2b^2+b^2c^2+c^2a^2)\geq(a^3b+a^3c+b^3a+b^3c+c^3a+c^3b)+(a^2bc+ab^2c+abc^2)$
This is equivalent to
$(a^2+b^2+c^2)(a^2+b^2+c^2-ab-bc-ca)\geq 0$
Which is true as $a^2+b^2+c^2-ab-bc-ca\geq 0\iff(a-b)^2+(b-c)^2+(c-a)^2\geq 0$

Second Solution (Popa Alexandru):
Since :

$$\frac{a^3-b^3}{a^2+ab+b^2}=a-b$$

and similars we get :

$$\sum_{cyc} \frac{a^3}{a^2+ab+b^2} = \sum_{cyc} \frac{b^3}{a^2+ab+b^2} = \frac{1}{2} \sum_{cyc} \frac{a^3+b^3}{a^2+ab+b^2}.$$

Now it remains to prove :

$$\frac{1}{2} \cdot \frac{a^3+b^3}{a^2+ab+b^2} \geq \frac{a+b}{6}$$

which is trivial .

Third Solution (Popa Alexandru): We have :

$$\sum_{cyc} \frac{a^3}{a^2+ab+b^2} = \sum_{cyc} \frac{b^3}{a^2+ab+b^2} = \frac{1}{2} \sum_{cyc} \frac{a^3+b^3}{a^2+ab+b^2}$$

Then we need to prove :

$$\frac{1}{2} \sum_{cyc} \frac{a^3+b^3}{a^2+ab+b^2} \geq \frac{a+b+c}{3}$$

We have :

$$\frac{1}{2} \sum_{cyc} \frac{a^3+b^3}{a^2+ab+b^2} - \frac{a+b+c}{3} = \frac{1}{2} \sum_{cyc} \left(\frac{a^3+b^3}{a^2+ab+b^2} - \frac{a+b}{3} \right) =$$

$$\frac{1}{2} \sum_{cyc} \frac{3a^3+3b^3-a^3-a^2b-ab^2-b^3-a^2b-ab^2}{3(a^2+ab+b^2)} = \frac{1}{2} \sum_{cyc} \frac{2(a-b)^2(a+b)}{3(a^2+ab+b^2)} \geq 0$$

Fourth Solution (Mohamed El-Alami): We have :

$$\sum_{cyc} \frac{a^3}{a^2+ab+b^2} = \sum_{cyc} \frac{b^3}{a^2+ab+b^2} = \frac{1}{2} \sum_{cyc} \frac{a^3+b^3}{a^2+ab+b^2}$$

Since $2(a^2+ab+b^2) \leq 3(a^2+b^2)$ we have :

$$LHS = \sum_{cyc} \frac{a^3+b^3}{2(a^2+ab+b^2)} \geq \sum_{cyc} \frac{a^3+b^3}{3(a^2+b^2)} \geq \frac{a+b+c}{3}$$

Problem 30 *'—'* (dgreenb801): Given $ab+bc+ca=1$ Show that:

$$\frac{\sqrt{3a^2+b^2}}{ab} + \frac{\sqrt{3b^2+c^2}}{bc} + \frac{\sqrt{3c^2+a^2}}{ca} \geq 6\sqrt{3}$$

Solution (Hoang Quoc Viet):
Let's make use of Cauchy-Schwarz as demonstrated as follows

$$\frac{\sqrt{(3a^2+b^2)(3+1)}}{2ab} \geq \frac{3a+b}{2ab}$$

Thus, we have the following estimations

$$\sum_{cyc} \frac{\sqrt{3a^2+b^2}}{ab} \geq 2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Finally, we got to prove that

$$\sum_{cyc} \frac{1}{a} \geq 3\sqrt{3}$$

However, from the given condition, we derive

$$abc \leq \frac{1}{3\sqrt{3}}$$

and

$$\sum_{cyc} \frac{1}{a} \geq 3\sqrt[3]{\frac{1}{abc}} \geq 3\sqrt{3}$$

Problem 31 *'Komal Magazine'* (Hoang Quoc Viet): Let a, b, c be real numbers. Prove that the following inequality holds

$$(a^2+2)(b^2+2)(c^2+2) \geq 3(a+b+c)^2$$

Solution (Popa Alexandru): Cauchy-Schwartz gives :

$$(a^2+2)(b^2+2) = (a^2+1)(1+b^2)+a^2+b^2+3 \geq (a+b)^2 + \frac{1}{2}(a+b)^2+3 = \frac{3}{2}((a+b)^2+2)$$

And Cauchy-Schwartz again

$$(a^2+2)(b^2+2)(c^2+2) \geq \frac{3}{2}((a+b)^2+2)(2+c^2) \geq \frac{3}{2}(\sqrt{2}(a+b)+\sqrt{2}c)^2 = RHS$$

Problem 32 *'mateforum.ro'* (Popa Alexandru): Let $a, b, c \geq 0$ and $a+b+c=1$. Prove that :

$$\frac{a}{\sqrt{b^2+3c}} + \frac{b}{\sqrt{c^2+3a}} + \frac{c}{\sqrt{a^2+3b}} \geq \frac{1}{\sqrt{1+3abc}}$$

Solution (Endrit Fejzullahu):
Using Holder's inequality
$$\left(\sum_{cyc} \frac{a}{\sqrt{b^2+3c}}\right)^2 \cdot \sum_{cyc} a(b^2+3c) \geq (a+b+c)^3 = 1$$
It is enough to prove that
$$1+3abc \geq \sum_{cyc} a(b^2+3c)$$
Homogenise $(a+b+c)^3 = 1$,

Also after Homogenising $\sum_{cyc} a(b^2+3c) = a^2b + b^2c + c^2a + 9abc + 3\sum_{sym} a^2b$
$(a+b+c)^3 = a^3+b^3+c^3+6abc+3\sum_{sym} a^2b$
It is enough to prove that
$a^3+b^3+c^3 \geq a^2b+b^2c+c^2a$
By AM-GM
$a^3+a^3+b^3 \geq 3a^2b$
$b^3+b^3+c^3 \geq 3b^2c$
$c^3+c^3+a^3 \geq 3c^2a$
Then $a^3+b^3+c^3 \geq a^2b+b^2c+c^2a$,done

Problem 33 *'Apartim De'* (Apartim De): If a, b, c, d be positive reals then prove that:

$$\frac{a^2+b^2}{ab+b^2} + \frac{b^2+c^2}{bc+c^2} + \frac{c^2+d^2}{cd+d^2} \geq \sqrt[3]{\frac{54a}{(a+d)}}$$

Solution (Aravind Srinivas):
Write the LHS as $\dfrac{(\frac{a}{b})^2+1}{(\frac{a}{b})+1}$+ two other similar termsfeel lazy to write them down. This is a beautiful application of Jensen's as the function for positve real t such that $f(t) = \dfrac{t^2+1}{t+1}$ is convex since $\left(\dfrac{t-1}{t+1}\right)^2 \geq 0$.
Thus, we get that $LHS \geq \dfrac{\left(\frac{(\sum \frac{a}{b} - \frac{d}{a}}{3}\right)^2+1}{\sum \frac{a}{b} - \frac{d}{a} + 3}$
I would like to write $\dfrac{a}{b} + \dfrac{b}{c} + \dfrac{c}{d} = K$ for my convenience with latexing.
so we have $\dfrac{\left((\frac{K}{3})^2+1\right)}{K+3} = \dfrac{K+\frac{9}{K}}{1+\frac{3}{K}} \geq \dfrac{6}{1+\frac{3}{K}}$
This is from $K + \dfrac{9}{K} \geq 6$ by AM GM.
Now $K = \sum \dfrac{a}{b} - \dfrac{d}{a} \geq 3\left(\dfrac{d}{a}\right)^{\frac{1}{3}}$ by AM GM.
Thus, we have $\dfrac{6}{1+\frac{3}{K}} \geq \dfrac{6a^{\frac{1}{3}}}{a^{\frac{1}{3}}+d^{\frac{1}{3}}} \geq \dfrac{3a^{\frac{1}{3}}}{\frac{a+d}{2}^{\frac{1}{3}}} = \left(\dfrac{54a}{a+d}\right)^{\frac{1}{3}}$

Problem 34 '—' (Raghav Grover): If a and b are non negative real numbers such that $a \geq b$. Prove that

$$a + \frac{1}{b(a-b)} \geq 3$$

First Solution (Dimitris Charisis):
If $a \geq 3$ the problem is obviously true.
Now for $a < 3$ we have :

$a^2b - ab^2 - 3ab + 3b^2 + 1 \geq 0 \Longleftrightarrow ba^2 - (b^2 + 3b)a + 3b^2 + 1 \geq 0$
It suffices to prove that $D \leq 0 \Longleftrightarrow b^4 + 6b^3 + 9b^2 - 12b^3 - 4b \Longleftrightarrow b(b-1)^2(b-4) \leq 0$ which is true.

Second Solution (dgreenb801):
$a + \dfrac{1}{b(a-b)} = b + (a-b) + \dfrac{1}{b(a-b)} \geq 3$ by AM-GM

Third Solution (geniusbliss):
since $a \geq b$ we have $\frac{a}{2} \geq \sqrt{b(a-b)}$ square this and substitute in this denominator we get,
$\dfrac{a}{2} + \dfrac{a}{2} + \dfrac{4}{a^2} \geq 3$ by AM-GM so done.

Problem 35 *'Vasile Cirtoaje'* (Dimitris Charisis):

$$(a^2 - bc)\sqrt{b+c} + (b^2 - ca)\sqrt{c+a} + (c^2 - ab)\sqrt{a+b} \geq 0$$

First Solution (Popa Alexandru): Denote $\frac{a+b}{2} = x^2, \ldots$, then the inequality becomes :

$$\sum_{cyc} xy(x^3 + y^3) \geq \sum_{cyc} x^2y^2(x+y)$$

which is equivalent with :

$$\sum_{cyc} xy(x+y)(x-y)^2 \geq 0$$

Second Solution (Popa Alexandru): Set $A = \sqrt{b+c}$ and similars . Therefore the inequality rewrites to :

$$A(a^2 - bc) + B(b^2 - ca) + C(c^2 - ab) \geq 0$$

We have that :

$$2\sum_{cyc} A(a^2 - bc) = \sum_{cyc} A[(a-b)(a+c) + (a-c)(a+b)] =$$

$$= \sum_{cyc} A(a-b)(a+c) + \sum_{cyc} B(b-a)(b+c) = \sum_{cyc} (a-b)[A(a+c) - B(b+c)] =$$

$$= \sum_{cyc} (a-b)\frac{A^2(a+c)^2 - B^2(b+c)^2}{A(a+c) + B(b+c)} = \sum_{cyc} \frac{(a-b)^2(a+c)(b+c)}{A(a+c) + B(b+c)} \geq 0$$

Problem 36 *'Cezar Lupu'* (Popa Alexandru): Let $a, b, c > 0$ such that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \sqrt{abc}$. Prove that:

$$abc \geq \sqrt{3(a+b+c)}$$

Solution (Endrit Fejzullahu):
$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\sqrt{abc} \iff ab+bc+ca=abc\sqrt{abc}$
By Am-Gm
$(ab+bc+ca)^2 \geq 3abc(a+b+c)$
Since ab + bc + ca = abc $\sqrt{abc}, wehave$
(abc) $^3 \geq 3abc(a+b+c) \implies abc \geq \sqrt{3(a+b+c)}$

Problem 37 *'Pham Kim Hung'* (Endrit Fejzullahu): Let a,b,c,d be positive real numbers satisfying $a+b+c+d=4$.Prove that

$$\frac{1}{11+a^2}+\frac{1}{11+b^2}+\frac{1}{11+c^2}+\frac{1}{11+d^2} \leq \frac{1}{3}$$

First Solution (Apartim De):
$f(x)=\dfrac{1}{11+x^2} \Leftrightarrow f''(x)=\dfrac{6(x^2-\frac{11}{3})}{(11+x^2)^3}$
$(x^2-\dfrac{11}{3})=\left(x-\sqrt{\dfrac{11}{3}}\right)\left(x+\sqrt{\dfrac{11}{3}}\right)$
If x $\in \left(-\sqrt{\dfrac{11}{3}},\sqrt{\dfrac{11}{3}}\right)$, $f''(x)<0$

Thus within the interval $\left(-\sqrt{\dfrac{11}{3}},\sqrt{\dfrac{11}{3}}\right)$, the quadratic polynomial is negative

thereby making $f''(x)<0,$ and thus $f(x)$ is concave within $\left(-\sqrt{\dfrac{11}{3}},\sqrt{\dfrac{11}{3}}\right)$.

Let $a \leq b \leq c \leq d$. If all of $a,b,c,d \in \left(0,\sqrt{\dfrac{11}{3}}\right)$,

Then by Jensen, $f(a)+f(b)+f(c)+f(d) \leq 4f\left(\dfrac{a+b+c+d}{4}\right)=4f(1)=\dfrac{4}{12}=\dfrac{1}{3}$

$f'(x)=\dfrac{-2x}{(11+x^2)^3}<0$ (for all positive x)

At most 2 of a,b,c,d(namely c& d) can be greater than $\sqrt{\dfrac{11}{3}}$

In that case,

$f(a)+f(b)+f(c)+f(d)<f(a-1)+f(b-1)+f(c-3)+f(d-3)<4f\left(\dfrac{a+b+c+d-8}{4}\right)=4f(-1)=\dfrac{4}{12}=\dfrac{1}{3}$

QED

Second Solution (Popa Alexandru): Rewrite the inequality in the following form

$$\sum_{cyc}\left(\frac{1}{11+a^2}-\frac{1}{12}\right) \geq 0$$

or equivalently

$$\sum_{cyc}(1-a)\cdot\frac{a+1}{a^2+11} \geq 0$$

Notice that if (a, b, c, d) is arranged in an increasing order then

$$\frac{a+1}{a^2+11} \geq \frac{b+1}{b^2+11} \geq \frac{c+1}{c^2+11} \geq \frac{d+1}{d^2+11}$$

The desired results follows immediately from the Chebyshev inequality.

Problem 38 *'Crux Mathematicorum'* (Apartim De): Let R, r, s be the circumradius, inradius, and semiperimeter, respectively, of an acute-angled triangle. Prove or disprove that

$$s^2 \geq 2R^2 + 8Rr + 3r^2$$

.
When does equality occur?

Solution (Virgil Nicula):

Proof. I"ll use the remarkable linear-angled identities $\left\|\begin{array}{cc} a^2+b^2+c^2 = 2\cdot\left(p^2-r^2-4Rr\right) & (1) \\ 4S = \left(b^2+c^2-a^2\right)\cdot\tan A & (2) \\ \sin 2A + \sin 2B + \sin 2C = \frac{2S}{R^2} & (3) \\ \cos A + \cos B + \cos C = 1 + \frac{r}{R} & (4) \end{array}\right\|$. Therefore,

$$\sum a^2 = \sum\left(b^2+c^2-a^2\right) \overset{(2)}{=} 4S\cdot\sum\frac{\cos A}{\sin A} = 8S\cdot\sum\frac{\cos^2 A}{\sin 2A} \overset{C.B.S.}{\geq} 8S\cdot\frac{\left(\sum\cos A\right)^2}{\sum\sin 2A} \overset{(3)\wedge(4)}{=} 8S\cdot\frac{\left(1+\frac{r}{R}\right)^2}{\frac{2S}{R^2}} =$$
$$= 4(R+r)^2 \Longrightarrow \boxed{a^2+b^2+c^2 \geq 4(R+r)^2} \overset{(1)}{\Longrightarrow} 2\cdot\left(p^2-r^2-4Rr\right) \geq 4(R+r)^2 \Longrightarrow \boxed{p^2 \geq 2R^2+8Rr+3r^2}.$$

Problem 39 *'Russia 1978'* (Endrit Fejzullahu): Let $0 < a < b$ and $x_i \in [a, b]$.Prove that

$$(x_1 + x_2 + ... + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_n}\right) \leq \frac{n^2(a+b)^2}{4ab}$$

Solution (Popa Alexandru):
We will prove that if $a_1, a_2, \ldots, a_n \in [a, b] (0 < a < b)$ then

$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \leq \frac{(a+b)^2}{4ab}n^2$$

$$P = (a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) = \left(\frac{a_1}{c} + \frac{a_2}{c} + \cdots + \frac{a_n}{c}\right)\left(\frac{c}{a_1} + \frac{c}{a_2} + \cdots + \frac{c}{a_n}\right) \leq$$
$$\leq \frac{1}{4}\left(\frac{a_1}{c} + \frac{c}{a_1} + \frac{a_2}{c} + \frac{c}{a_2} + \cdots + \frac{a_n}{c} + \frac{c}{a_n}\right)^2$$

Function $f(t) = \frac{c}{t} + \frac{t}{c}$ have its maximum on $[a, b]$ in a or b. We will choose c such that $f(a) = f(b), c = \sqrt{ab}$.
Then $f(t) \leq \sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}}$. Then

$$P \leq n^2 \left(\sqrt{\frac{a}{b}} + \sqrt{b\frac{b}{a}} \right)^2 \cdot \frac{1}{4} = n^2 \frac{(a+b)^2}{4ab}$$

Problem 40 *'Vasile Cartoaje'* (keyree10): a, b, c are non-negative reals. Prove that

$$81abc \cdot (a^2 + b^2 + c^2) \leq (a + b + c)^5$$

Solution (Popa Alexandru):

$$81abc(a^2 + b^2 + c^2) \leq 27\frac{(ab + bc + ca)^2}{a + b + c}(a^2 + b^2 + c^2) \leq (a + b + c)^5$$

By p, q, r the last one is equivalent with :

$$p^6 - 27q^2p^2 + 54q^3 \geq 0 \Leftrightarrow (p^2 - 3q)^2 \geq 0$$

Problem 41 *'mateforum.ro'* (Popa Alexandru): Let $a, b, c > 0$ such that $a^3 + b^3 + 3c = 5$. Prove that :

$$\sqrt{\frac{a+b}{2c}} + \sqrt{\frac{b+c}{2a}} + \sqrt{\frac{c+a}{2b}} \leq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

Solution (Sayan Mukherjee):
$a^3 + 1 + 1 + b^3 + 1 + 1 + 3c = 9 \geq 3a + 3b + 3c \implies a + b + c \leq 3$(AM-GM)
Also $abc \leq \left(\frac{a+b+c}{3}\right) = 1$(AM-GM)
Applying CS on the LHS;
$\left[\sum \sqrt{\frac{a+b}{2c}}\right]^2 \leq (a + b + c)\left(\sum \frac{1}{a}\right)$
It is left to prove that $\sum \frac{1}{a} \geq \sum a$
But this $\implies \sum \frac{1}{a} \geq 3$
Which is perfextly true, as from AM-GM on the LHS;
$\sum \frac{1}{a} \geq 3\sqrt[3]{\frac{1}{abc}} \geq 3\,[\because abc \leq 1]$

Problem 42 *'Sayan Mukherjee'* (Sayan Mukherjee): Let $a, b, c > 0$ PT: If a, b, c satisfy $\sum \frac{1}{a^2+1} = \frac{1}{2}$ Then we always have:

$$\sum \frac{1}{a^3+2} \leq \frac{1}{3}$$

Solution (Endrit Fejzullahu):

$$a^3 + a^3 + 1 \geq 3a^2 \implies a^3 + a^3 + 1 + 3 \geq 3a^2 + 3 \implies \frac{1}{3(a^2+1)} \geq \frac{1}{2(a^3+2)} \implies \frac{1}{3} \geq \sum \frac{1}{a^3+2}$$

Problem 43 *'Russia 2002'* (Endrit Fejzullahu): Let a, b, c be positive real numbers with sum 3.Prove that

$$\sqrt{a} + \sqrt{b} + \sqrt{c} \geq ab + bc + ca$$

Solution (Sayan Mukherjee):
$2(ab + bc + ca) = 9 - (a^2 + b^2 + c^2)$
as $a + b + c = 9$. Hence the inequality
$\implies 2\sum \sqrt{a} \geq 9 - (a^2 + b^2 + c^2)$
$\implies \sum (a^2 + \sqrt{a} + \sqrt{a}) \geq 9$
Perfectly true from AGM as :
$a^2 + \sqrt{a} + \sqrt{a} \geq 3a$

Problem 44 *'India 2002'* (Raghav Grover): For any natural number n prove that

$$\frac{1}{2} \leq \frac{1}{n^2+1} + \frac{1}{n^2+2} + ... + \frac{1}{n^2+n} \leq \frac{1}{2} + \frac{1}{2n}$$

Solution (Sayan Mukherjee):
$\sum_{k=1}^{n} \frac{k}{n^2+k} = \sum_{k=1}^{n} \frac{k^2}{n^2k+k^2} \geq \frac{\sum_{k=1}^{n} k}{n^2 \sum_{k=1}^{n} k + \sum_{k=1}^{n} k^2} = \frac{3(n^2+n)}{2(3n^2+2n+1)} > \frac{1}{2}$
As it is equivalent to :$3n^2 + 3n > 3n^2 + 2n + 1 \implies n > 1$
For the 2nd part;
$\sum \frac{k}{n^2+k} = \sum 1 - \frac{n^2}{n^2+k} = n - n^2 \sum \frac{1}{n^2+k}$ and then use AM-HM for $\sum \frac{1}{n^2+k}$ So we get the desired result.

Problem 45 *'—'* (Sayan Mukherjee): For $a, b, c > 0; a^2 + b^2 + c^2 = 1$ find P_{min} if:

$$P = \sum_{cyc} \frac{a^2b^2}{c^2}$$

Solution (Endrit Fejzullahu):
Let $x = \dfrac{ab}{c}, y = \dfrac{bc}{a} and z = \dfrac{ca}{b}$
Then Obviously By Am-Gm
$x^2 + y^2 + z^2 \geq xy + yz + zx = a^2 + b^2 + c^2 = 1$,then $P_{min} = 1$

Problem 46 *'Pham Kim Hung'* (Endrit Fejzullahu): Let a, b, c be positive real numbers such that $a+b+c = 3$ .Prove that

$$\frac{a^2}{a + 2b^2} + \frac{b^2}{b + 2c^2} + \frac{c^2}{c + 2a^2} \geq 1$$

Solution (Popa Alexandru): We start with a nice use of AM-GM :

$$\frac{a^2}{a + 2b^2} = \frac{a^2 + 2ab^2 - 2ab^2}{a + 2b^2} = \frac{a(a + 2b^2)}{a + 2b^2} - \frac{2ab^2}{a + 2b^2} \geq a - \frac{2}{3}\sqrt[3]{a^2b^2}$$

Suming the similars we need to prove :

$$\sqrt[3]{a^2b^2} + \sqrt[3]{b^2c^2} + \sqrt[3]{c^2a^2} \leq 3$$

By AM-GM :

$$\sum_{cyc} \sqrt[3]{a^2b^2} \leq \sum_{cyc} \frac{2ab + 1}{3} = \frac{1}{3}\left(2\sum_{cyc} ab + 3\right) \leq 3 \Leftrightarrow ab + bc + ca \leq 3 \Leftrightarrow 3(ab + bc + ca) \leq (a + b + c)^2$$

Problem 47 *'mateforum.ro'* (Popa Alexandru): Let $a, b, c > 0$ such that $a + b + c \leq \frac{3}{2}$. Prove that :

$$\frac{a}{b + c} + \frac{b}{c + a} + \frac{c}{a + b} \leq \frac{a}{(b + c)^2} + \frac{b}{(c + a)^2} + \frac{c}{(a + b)^2}$$

Solution:
WLOG $a \geq b \geq c$
Then By Chebyshev's inequality we have
$RHS \geq \dfrac{1}{3} \cdot LHS \cdot \left(\dfrac{1}{a + b} + \dfrac{1}{b + c} + \dfrac{1}{c + a}\right)$
It is enough to show that
$\dfrac{1}{a + b} + \dfrac{1}{b + c} + \dfrac{1}{c + a} \geq 3$
By Cauchy Schwartz

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\geq\frac{9}{2(a+b+c)}\geq 3 \iff a+b+c\leq\frac{3}{2}, \text{ done}$$

Problem 48 *'USAMO 2003'* (Hassan Al-Sibyani): Let a,b,c be positive real numbers. Prove that:

$$\frac{(2a+b+c)^2}{2a^2+(b+c)^2}+\frac{(2b+a+c)^2}{2b^2+(a+c)^2}+\frac{(2c+a+b)^2}{2c^2+(a+b)^2}\leq 8$$

Solution (Popa Alexandru):
Suppose $a+b+c=3$
The inequality is equivalent with :

$$\frac{(a+3)^2}{2a^2+(3-a)^2}+\frac{(b+3)^2}{2b^2+(3-b)^2}+\frac{(c+3)^2}{2c^2+(3-c)^2}\leq 8$$

For this we prove :

$$\frac{(a+3)^2}{2a^2+(3-a)^2}\leq \frac{4}{3}a+\frac{4}{3}\Leftrightarrow 3(4a+3)(a-1)^2\geq 0$$

done.

Problem 49 *'Marius Mainea'* (Popa Alexandru): Let $x,y,z>0$ such that $x+y+z=xyz$. Prove that :

$$\frac{x+y}{1+z^2}+\frac{y+z}{1+x^2}+\frac{z+x}{1+y^2}\geq\frac{27}{2xyz}$$

Solution (socrates): Using Cauchy-Schwarz and AM-GM inequality we have:

$$\frac{x+y}{1+z^2}+\frac{y+z}{1+x^2}+\frac{z+x}{1+y^2}=\frac{(x+y)^2}{x+y+(x+y)z^2}+\frac{(y+z)^2}{y+z+(y+z)x^2}+\frac{(z+x)^2}{z+x+(z+x)y^2}\geq\frac{((x+y)+(y+z)+(z+x))^2}{2(x+y+z)+\sum x^2(y+z)}=$$

$$\frac{4(x+y+z)^2}{2xyz+\sum x^2(y+z)}=\frac{4(x+y+z)^2}{\prod(x+y)}\geq 4(x+y+z)^2\cdot\frac{27}{8}\frac{1}{(x+y+z)^3}=\frac{27}{2xyz}$$

Problem 50 '—' (socrates): Let $a,b,c>0$ such that $ab+bc+ca=1$. Prove that :

$$abc(a+\sqrt{a^2+1})(b+\sqrt{b^2+1})(c+\sqrt{c^2+1})\leq 1$$

Solution (dgreenb801): Note that $\sqrt{a^2+1}=\sqrt{a^2+ab+bc+ca}=\sqrt{(a+b)(a+c)}$
Also, by Cauchy, $(a+\sqrt{(a+b)(a+c)})^2\leq(a+(a+b))(a+(a+c))=(2a+b)(2a+c)$
So after squaring both sides of the inequality, we have to show

$$1=(ab+bc+ca)^6\geq a^2b^2c^2(2a+b)(2a+c)(2b+a)(2b+c)(2c+a)(2c+b)=$$

$$(2ac+bc)(2ab+bc)(2bc+ac)(2ab+ac)(2cb+ab)(2ca+ab)$$

which is true by AM-GM.

Problem 51 *'Asian Pacific Mathematics Olympiad 2005'* (dgreenb801): Let $a, b, c > 0$ such that $abc = 8$. Prove that:

$$\frac{a^2}{\sqrt{(1+a^3)(1+b^3)}}+\frac{b^2}{\sqrt{(1+b^3)(1+c^3)}}+\frac{c^2}{\sqrt{(1+c^3)(1+a^3)}} \geq \frac{4}{3}$$

Solution (Popa Alexandru): By AM-GM :

$$\sqrt{a^3+1} \leq \frac{a^2+2}{2}.$$

Then we have :

$$\sum_{cyc} \frac{a^2}{\sqrt{(a^3+1)(b^3+1)}} \geq 4 \sum_{cyc} \frac{a^2}{(a^2+2)(b^2+2)}$$

So we need to prove

$$\sum_{cyc} \frac{a^2}{(a^2+2)(b^2+2)} \geq \frac{1}{3}.$$

which is equivalent with :

$$a^2b^2+b^2c^2+c^2a^2+2(a^2+b^2+c^2) \geq 72.$$

, true by AM-GM

Problem 52 *'Lucian Petrescu'* (Popa Alexandru): Prove that in any acute-angled triangle ABC we have :

$$\frac{a+b}{\cos C}+\frac{b+c}{\cos A}+\frac{c+a}{\cos B} \geq 4(a+b+c)$$

First Solution (socrates): The inequality can be rewritten as

$$\frac{ab(a+b)}{a^2+b^2-c^2}+\frac{bc(b+c)}{b^2+c^2-a^2}+\frac{ca(c+a)}{c^2+a^2-b^2} \geq 2(a+b+c)$$

or

$$\sum_{cyclic} a\left(\frac{b^2}{a^2+b^2-c^2}+\frac{c^2}{c^2+a^2-b^2}\right) \geq 2(a+b+c)$$

Applying Cauchy Schwarz inequality we get

$$\sum_{cyclic} a\left(\frac{b^2}{a^2+b^2-c^2}+\frac{c^2}{c^2+a^2-b^2}\right) \geq \sum a \frac{(b+c)^2}{(a^2+b^2-c^2)+(c^2+a^2-b^2)}$$

or

$$\sum_{cyclic} a\left(\frac{b^2}{a^2+b^2-c^2}+\frac{c^2}{c^2+a^2-b^2}\right) \geq \sum \frac{(b+c)^2}{2a}$$

So, it is enough to prove that

$$\sum \frac{(b+c)^2}{2a} \geq 2(a+b+c)$$

which is just CS as above.

Second Solution (Mateescu Constantin):
From the law of cosinus we have $a = b\cos C + c\cos B$and $c = b\cos A + a\cos B$.
$\Longrightarrow \dfrac{a+c}{\cos B} = a + c + \dfrac{b(\cos A + \cos C)}{\cos B} = a + c + 2R\tan B(\cos A + \cos C)$. Then:
$\sum \dfrac{a+c}{\cos B} = 2(a+b+c) + 2R\sum \tan B \cdot (\cos A + \cos C)$ and the inequality becomes:
$2R\sum \tan B(\cos A + \cos C) \geq 2\sum a = 4R\sum \sin A$
$\Longleftrightarrow \sum \tan B\cos A + \sum \tan B\cos C \geq 2\sum \sin A = 2\sum \tan A\cos A$
Wlog assume that $A \leq B \leq C$. Then $\cos A \geq \cos B \geq \cos C$ and $\tan A \leq \tan B \leq \tan C$, so
$\sum \tan A\cos A \leq \sum \tan B\cos A$ and $\sum \tan A\cos A \leq \sum \tan B\cos C$, according to rearrangement inequality
.
Adding up these 2 inequalities yields the conclusion.

Third Solution (Popa Alexandru):

$$LHS \geq \frac{(2(a+b+c))^2}{(a+b)\cos C + (b+c)\cos A + (c+a)\cos B} = 4(a+b+c)$$

Problem 53 *'—'* (socrates): Given $x_1, x_2, .., x_n > 0$ such that $\sum_{i=1}^{n} x_i = 1$, prove that

$$\sum_{i=1}^{n} \frac{x_i + n}{1 + x_i^2} \leq n^2$$

Solution (Hoang Quoc Viet):
Without loss of generality, we may assume that

$$x_1 \geq x_2 \geq \cdots \geq x_n$$

Therefore, we have

$$nx_1 - 1 \geq nx_2 - 1 \geq \cdots \geq nx_n - 1$$

and

$$\frac{x_1}{x_1^2+1} \geq \frac{x_2}{x_2^2+1} \geq \cdots \geq \frac{x_n}{x_n^2+1}$$

Hence, by Chebyshev inequality, we get

$$\sum_{i=1}^{n} \frac{(nx_i - 1)x_i}{x_i^2+1} \geq \frac{1}{n}\left[n\left(\sum_{i=1}^{n} x_i\right) - n\right]\left(\sum_{i=1}^{n} \frac{x_i}{x_i^2+1}\right) = 0$$

Thus, we get the desired result, which is

$$\sum_{i=1}^{n} \frac{x_i + n}{1 + x_i^2} \leq n^2$$

Problem 54 *'Hoang Quoc Viet'* (Hoang Quoc Viet): Let a, b, c be positive reals satisfying $a^2 + b^2 + c^2 = 3$. Prove that

$$\frac{a^3}{2b^2 + c^2} + \frac{b^3}{2c^2 + a^2} + \frac{c^3}{2a^2 + b^2} \geq 1$$

Solution (Hoang Quoc Viet):
Using Cauchy Schwartz, we get

$$\sum_{cyc} \frac{a^3}{2b^2 + c^2} \geq \frac{(a^2 + b^2 + c^2)^2}{2\sum_{cyc} ab^2 + \sum_{cyc} ac^2}$$

Hence, it suffices to prove that

$$ab^2 + bc^2 + ca^2 \leq 3$$

and

$$a^2b + b^2c + c^2a \leq 3$$

However, applying Cauchy Schwartz again, we obtain

$$a(ab) + b(bc) + c(ca) \leq \sqrt{(a^2 + b^2 + c^2)\left((ab)^2 + (bc)^2 + (ca)^2\right)}$$

In addition to that, we have

$$(ab)^2 + (bc)^2 + (ca)^2 \leq \frac{(a^2 + b^2 + c^2)^2}{3}$$

Hence, we complete our proof.

Problem 55 *'Iran 1998'* (saif): Let $x, y, z > 1$ such that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$ prove that:

$$\sqrt{x + y + z} \geq \sqrt{x - 1} + \sqrt{x - 1} + \sqrt{z - 1}$$

First Solution (beautifulliar):
Note that you can substitute $\sqrt{x-1} = a, \sqrt{y-1} = b, \sqrt{z-1} = c$ then you need to prove that $\sqrt{a^2 + b^2 + c^2 + 3} \geq a + b + c$ while you have $\dfrac{1}{a^2+1} + \dfrac{1}{b^2+1} + \dfrac{1}{c^2+1} = 2$ or equivalently $a^2b^2 + b^2c^2 + c^2a^2 + 2a^2b^2c^2 = 1$. next, substitute $ab = \cos x, bc = \cos y, ca = \cos z$ where x, y, z are angles of triangle. since you need to prove that $\sqrt{a^2 + b^2 + c^2 + 3} \geq a + b + c$ then you only need to prove that $3 \geq 2(ab + bc + ca)$ or $\cos x + \cos B + \cos C \leq \dfrac{3}{2}$ which is trivial.

Second Solution (Sayan Mukherjee):
$\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 2 \implies \sum \dfrac{x-1}{x} = 1$
Hence $(x + y + z) \sum \dfrac{x-1}{x} \geq \left(\sum \sqrt{x-1}\right)^2$ (From CS)
$\implies \sqrt{x + y + z} \geq \sum \sqrt{x-1}$

Problem 56 *'IMO 1998'* (saif): Let $a_1, a_2, ..., a_n > 0$ such that $a_1 + a_2 + ... + a_n < 1$. prove that

$$\frac{a_1.a_2...a_n(1 - a_1 - a_2 - ... - a_n)}{(a_1 + a_2 + ... + a_n)(1 - a_1)(1 - a_2)...(1 - a_n)} \leq \frac{1}{n^{n-1}}$$

Solution (beautifulliar):
Let $a_{n+1} = 1 - a_1 - a_2 - \cdots - a_n$ the we arrive at $\frac{a_1a_2 \dots a_n a_{n+1}}{(1-a_1)(1-a_2)\dots(1-a_n)} \leq \frac{1}{n^{n+1}}$(it should be n^{n+1} right?) which follows from am-gm $a_1 + a_2 + \cdots + a_n \geq n\sqrt[n]{a_1a_2 \dots a_n}, a_2 + a_3 + \cdots + a_{n+1} \geq n\sqrt[n]{a_2a_3 \dots a_{n+1}}, a_1 + a_3 + \cdots + a_{n+1} \geq n\sqrt[n]{a_1a_3 \dots a_{n+1}}$, and so on... you will find it easy.

Problem 57 *'—'* (beautifulliar): Let n be a positive integer. If $x_1, x_2, \dots, x_n$ are real numbers such that $x_1 + x_2 + \cdots + x_n = 0$ and also $y = \max\{x_1, x_2, \dots, x_n\}$ and also $z = \min\{x_1, x_2, \dots, x_n\}$, prove that

$$x_1^2 + x_2^2 + \cdots + x_n^2 + nyz \leq 0$$

Solution (socrates):
$(x_i - y)(x_i - z) \leq 0 \ \forall i = 1, 2, ..., n$ so $\sum_{i=1}^{n}(x_i^2 + yz) \leq \sum_{i=1}^{n}(y+z)x_i = 0$ and the conclusion follows.

Problem 58 *'Pham Kim Hung'* (Endrit Fejzullahu): Suppose that x, y, z are positive real numbers and $x^5 + y^5 + z^5 = 3$.Prove that

$$\frac{x^4}{y^3} + \frac{y^4}{z^3} + \frac{z^4}{x^3} \geq 3$$

Solution (Popa Alexandru):
By a nice use of AM-GM we have :

$$10\left(\frac{x^4}{y^3} + \frac{y^4}{z^3} + \frac{z^4}{x^3} + 3(x^5 + y^5 + z^5)^2 \geq 19(\sqrt[19]{x^{100}} + \sqrt[19]{y^{100}} + \sqrt[19]{z^{100}})\right.$$

So it remains to prove :

$$3 + 19(\sqrt[19]{x^{100}} + \sqrt[19]{y^{100}} + \sqrt[19]{z^{100}}) \geq 20(x^5 + y^5 + z^5)$$

which is true by AM-GM .

Problem 59 *'China 2003'* (bokagadha): x, y, and z are positive real numbers such that $x + y + z = xyz$. Find the minimum value of:

$$x^7(yz - 1) + y^7(xz - 1) + z^7(xy - 1)$$

Solution (Popa Alexandru):
By AM-GM and the condition you get

$$xyz \geq 3\sqrt{3}$$

Also observe that the condition is equivalent with

$$yz - 1 = \frac{y+z}{x}$$

So the

$$LHS = x^6(y+z) + y^6(z+x) + z^6(x+y) \geq 6\sqrt[6]{x^{14}y^{14}z^{14}} \geq 216\sqrt{3}$$

Problem 60 *'Austria 1990'* (Rofler):

$$\sqrt{2\sqrt{3\sqrt{4\sqrt{...\sqrt{N}}}}} < 3 - - - \forall N \in \mathbb{N}^{\geq 2}$$

Solution (Brut3Forc3): We prove the generalization $\sqrt{m\sqrt{(m+1)\sqrt{\ldots\sqrt{N}}}} < m+1$, for $m+2$. For $m = N$, this is equivalent to $\sqrt{N} < N+1$, which is clearly true. We now induct from $m = N$ down. Assume that $\sqrt{(k+1)\sqrt{(k+2)\ldots\sqrt{N}}} < k+2$. Multiplying by k and taking the square root gives $\sqrt{k\sqrt{(k+1)\sqrt{\ldots\sqrt{N}}}} < \sqrt{k(k+2)} < k+1$, completing the induction.

Problem 61 *'Tran Quoc Anh'* (Sayan Mukherjee): Given $a, b, c \geq 0$ Prove that:

$$\sum_{cyc} \sqrt{\frac{a+b}{b^2+4bc+c^2}} \geq \frac{3}{\sqrt{a+b+c}}$$

Solution (Hoang Quoc Viet): Using Cauchy inequality, we get

$$\sum_{cyc} \sqrt{\frac{a+b}{b^2+4bc+c^2}} \geq \sum_{cyc} \sqrt{\frac{2(a+b)}{3(b+c)^2}} \geq \sum_{cyc} 3\sqrt[6]{\frac{8}{27(a+b)(b+c)(c+a)}}$$

However, we have

$$(a+b)(b+c)(c+a) \leq \frac{8(a+b+c)^3}{27}$$

Hence, we complete our proof here.

Problem 62 *'Popa Alexandru'* (Popa Alexandru): Let $a, b, c > 0$ such that $a+b+c=1$. Show that :

$$\frac{a^2+ab}{1-a^2}+\frac{b^2+bc}{1-b^2}+\frac{c^2+ca}{1-c^2}\geq\frac{3}{4}$$

First Solution (Endrit Fejzullahu): Let $a+b=x$, $b+c=y$, $c+a=z$,given inequality becomes

$$\sum\frac{x}{y}+\sum\frac{x}{x+y}\geq\frac{9}{2}$$

Then By Cauchy Schwartz

$$LHS=\sum_{cyc}\frac{x}{y}+\sum_{cyc}\frac{x}{x+y}\geq\frac{(\sum x)^4}{(\sum xy)(\sum xy+\sum x^2)}\geq\frac{8(\sum x)^4}{(\sum xy+(\sum x)^2)^2}\geq\frac{9}{2}$$

Second Solution (Hoang Quoc Viet): Without too many technical terms, we have

$$\left(\sum_{cyc}\frac{x+y}{4y}+\sum_{cyc}\frac{x}{x+y}\right)\geq\sum_{cyc}\sqrt{\frac{x}{y}}\geq 3$$

Therefore, it is sufficient to check that

$$\sum_{cyc}\frac{x}{y}\geq 3$$

which is Cauchy inequality for 3 positive reals.

Problem 63 *'India 2007'* (Sayan Mukherjee): For positive reals a, b, c. Prove that:

$$(a+b+c)^2(ab+bc+ca)^2\leq 3(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ac+a^2)$$

First Solution (Endrit Fejzullahu): I've got an SOS representation of :

$$RHS-LHS=\frac{1}{2}((x+y+z)^2(x^2y^2+y^2z^2+z^2x^2-xyz(x+y+z))+(xy+yz+zx)(x^2+y^2+z^2-xy-yz-zx))$$

So I may assume that the inequality is true for reals

Second Solution (Popa Alexandru): Since

$$a^2+ab+b^2\geq\frac{3}{4}(a-b)^2\Leftrightarrow(a-b)^2\geq 0$$

It'll be enough to prove that :

$$\frac{81}{64}(a+b)^2(b+c)^2(c+a)^2\geq(a+b+c)^2(ab+bc+ca)^2$$

The last one is famous and very easy by AM-GM.

Problem 64 *'Popa Alexandru'* (Endrit Fejzullahu): Let $a, b, c > 0$ such that $(a+b)(b+c)(c+a) = 1$. Show that :

$$\frac{3}{16abc} \geq a+b+c \geq \frac{3}{2} \geq 12abc$$

First Solution (Apartim De): By AM-GM, $\frac{(a+b)(b+c)(c+a)}{8} \geq abc \Leftrightarrow \frac{1}{8} \geq abc \Leftrightarrow \frac{2}{3} \geq \frac{16}{3}abc$ By AM-GM,
$(a+b)+(b+c)+(c+a) \geq 3$
$\Leftrightarrow (a+b+c) \geq \frac{3}{2} > \frac{2}{3}$
Lemma:We have for any positive reals x, y, z and vectors $\overrightarrow{MA}, \overrightarrow{MB}, \overrightarrow{MC}$
$\left(x\overrightarrow{MA} + y\overrightarrow{MB} + z\overrightarrow{MC}\right)^2 \geq 0$
$\Leftrightarrow (x+y+z)(xMA^2 + yMB^2 + zMC^2) \geq \left(xyAB^2 + yzBC^2 + zxCA^2\right)$
Now taking $x = y = z = 1$ and M to be the circumcenter of the triangle with sides p, q, r such that $pqr = 1$, and the area of the triangle=Δ, we have by the above lemma,
$9R^2 \geq p^2 + q^2 + r^2 \geq 3\,(pqr)^{\frac{2}{3}} = 3 \Leftrightarrow 3R^2 \geq 1 \Leftrightarrow 16\mathbf{\Delta^2} \leq 3$
$\Leftrightarrow (p+q+r)(p+q-r)(q+r-p)(r+p-q) \leq 3$
Now plugging in the famous Ravi substitution i.e,
$p = (a+b); q = (b+c); r = (c+a)$
$\Leftrightarrow (a+b+c) \leq \frac{3}{16abc}$

Second Solution (Popa Alexandru): Remember that $8(a+b+c)(ab+bc+ca) \leq 9(a+b)(b+c)(c+a)$ and $a+b+c \geq \sqrt{3(ab+bc+ca)}$ So

$$\frac{9}{8} \geq (a+b+c)(ab+bc+ca) \geq (ab+bc+ca)\sqrt{3(ab+bc+ca)}$$

$$\Leftrightarrow \frac{81}{64} \geq 3(ab+bc+ca)^3 \Leftrightarrow ab+bc+ca \leq \frac{3}{4}$$

Now we use :

$$3abc(a+b+c) \leq (ab+bc+ca)^2 \leq \frac{9}{16} \Rightarrow \frac{3}{16abc} \geq a+b+c$$

For the second recall for :

$$(a+b+c)^3 \geq a^3+b^3+c^3+3(a+b)(b+c)(c+a)$$

So

$$(a+b+c)^3 \geq a^3+b^3+c^3+3$$

Now Muirhead shows that :

$$8(a^3+b^3+c^3) \geq 3(a+b)(b+c)(c+a) \Rightarrow a^3+b^3+c^3 \geq \frac{3}{8}$$

There we have :

$$(a+b+c)^3 \geq \frac{3}{8} + 3 \Leftrightarrow a+b+c \geq \frac{3}{2}$$

For the last one by AM-GM we have :

$$(a+b)(b+c)(c+a) \geq 8abc \Rightarrow 8abc \leq 1$$

and the conclusion follows , done.

Problem 65 *'IMO 1988 shortlist'* (Apartim De): In the plane of the acute angled triangle ΔABC, L is a line such that u, v, w are the lengths of the perpendiculars from A, B, C respectively to L. Prove that

$$u^2 \tan A + v^2 \tan B + w^2 \tan C \geq 2\Delta$$

where Δ is the area of the triangle.

Solution (Hassan Al-Sibyani): Consider a Cartesian system with the x-axis on the line BC and origin at the foot of the perpendicular from A to BC, so that A lies on the y-axis. Let A be $(0, \alpha), B(-\beta, 0), C(\gamma, 0)$, where $\alpha, \beta, \gamma > 0$ (because ABC is acute-angled). Then

$\tan B = \dfrac{\alpha}{\beta}$

$\tan C = \dfrac{\alpha}{\gamma}$

$\tan A = -\tan(B+C) = \dfrac{\alpha(\beta+\gamma)}{\alpha^2 - \beta\gamma}$

here $\tan A > 0$,so $\alpha^2 > \beta\gamma$. Let L have equation $x\cos\theta + y\sin\theta + p = 0$

Then

$u^2 \tan A + v^2 \tan B + w^2 \tan C$

$= \dfrac{\alpha(\beta+\gamma)}{\alpha^2-\beta\gamma}(\alpha\sin\theta + p)^2 + \dfrac{\alpha}{\beta}(-\beta\cos\theta + p)^2 + \dfrac{\alpha}{\gamma}(\gamma\cos\theta + p)^2$

$= \alpha^2\sin^2\theta + 2\alpha p\sin\theta + p^2)\dfrac{\alpha(\beta+\gamma)}{\alpha^2-\beta\gamma} + \alpha(\beta+\gamma)\cos^2\theta + \dfrac{\alpha(\beta+\gamma)}{\beta\gamma}p^2$

$= \dfrac{\alpha(\beta+\gamma)}{\beta\gamma(\alpha^2-\beta\gamma)}(\alpha^2p^2 + 2\alpha p\beta\gamma\sin\theta + \alpha^2\beta\gamma\sin^2\theta + \beta\gamma(\alpha^2-\beta\gamma)\cos^2\theta)$

$= \dfrac{\alpha(\beta+\gamma)}{\beta\gamma(\alpha^2-\beta\gamma)}[(\alpha p + \beta\gamma\sin\theta)^2 + \beta\gamma(\alpha^2-\beta\gamma)] \geq \alpha(\beta+\gamma) = 2\Delta$

with equality when $\alpha p + \beta\gamma\sin\theta = 0$, i.e., if and only if L passes through $(0, \beta\gamma/\alpha)$, which is the orthocenter of the triangle.

Problem 66 *'—'* (Hassan Al-Sibyani): For positive real number a, b, c such that $abc \leq 1$, Prove that:

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq a + b + c$$

Solution (Endrit Fejzullahu): It is easy to prove that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{a+b+c}{\sqrt[3]{abc}}$$

and since $abc \leq 1$ then $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq a + b + c$,as desired .

Problem 67 *'Endrit Fejzullahu'* (Endrit Fejzullahu): Let a, b, c, d be positive real numbers such that $a + b + c + d = 4$.Find the minimal value of :

$$\sum_{cyc} \frac{a^4}{(b+1)(c+1)(d+1)}$$

Solution (Sayan Mukherjee): From AM-GM,

$$P=\sum_{cyc}\left[\frac{a^4}{(b+1)(c+1)(d+1)}+\frac{b+1}{16}+\frac{c+1}{16}+\frac{d+1}{16}\right]\geq\sum_{cyc}\left[\frac{4a}{8}\right]$$

$$\Longrightarrow\sum_{cyc}\left[\frac{a^4}{(b+1)(c+1)(d+1)}\right]\geq\frac{a+b+c+d}{2}-\frac{3}{16}(a+b+c+d)-4\cdot\frac{3}{16}$$

$$\therefore\sum_{cyc}\left[\frac{a^4}{(b+1)(c+1)(d+1)}\right]\geq\frac{4}{2}-\frac{6}{4}=2-\frac{3}{2}=\frac{1}{2}$$

So $P_{min}=\dfrac{1}{2}$

Problem 68 *'Sayan Mukherjee'* (Sayan Mukherjee): Prove that

$$\sum_{cyc}\frac{x}{7+z^3+y^3}\leq\frac{1}{3}\forall 1\geq x,y,z\geq 0$$

Solution (Toang Huc Khein):
$\displaystyle\sum\frac{x}{7+y^3+z^3}\leq\sum\frac{x}{6+x^3+y^3+z^3}\leq\sum\frac{x}{3(x+y+z)}=\frac{1}{3}$ because $x^3+y^3+z^3+6\geq 3(x+y+z)\Leftrightarrow$ $\displaystyle\sum(x-1)^2(x+2)\geq 0$

Problem 69 *'Marius Mainea'* (Toang Huc Khein): Let $x,y,z>0$ with $x+y+z=1$. Then :

$$\frac{x^2-yz}{x^2+x}+\frac{y^2-zx}{y^2+y}+\frac{z^2-xy}{z^2+z}\leq 0$$

First Solution (Endrit Fejzullahu):

$$\frac{x^2-yz}{x^2+x}=1-\frac{x+yz}{x^2+x}$$

Inequality is equivalent with :

$$\sum_{cyc}\frac{x+yz}{x^2+x}\geq 3$$

$$\sum_{cyc}\frac{x+yz}{x^2+x}=\sum_{cyc}\frac{1}{x+1}+\sum_{cyc}\frac{yz}{x^2+x}$$

By Cauchy-Schwarz inequality

$$\sum_{cyc}\frac{1}{x+1}\geq\frac{9}{4}$$

And

$$\sum_{cyc} \frac{yz}{x^2+x} \geq \frac{(xy+yz+zx)^2}{x^2yz+y^2xz+z^2xy+3xyz} = \frac{(xy+yz+zx)^2}{4xyz} \geq \frac{3}{4} \iff (xy+yz+zx)^2 \geq 3xyz$$

This is true since $x+y+z=1$ and $(xy+yz+zx)^2 \geq 3xyz(x+y+z)$

Second Solution (Endrit Fejzullahu): Let us observe that by Cauchy-Schwartz we have :

$$\sum_{cyc} \frac{x+yz}{x^2+x} = \sum_{cyc} \frac{x(x+y+z)+yz}{x(x+1)} = \sum_{cyc} \frac{(x+y)(x+z)}{x(x+y+x+z)} = \sum_{cyc} \frac{1}{\frac{x}{x+y}+\frac{x}{x+z}} \geq \frac{(1+1+1)^2}{3} = \frac{9}{3} = 3$$

Then we can conclude that :

$$LHS = \sum_{cyc} \frac{x^2+x}{x^2+x} - \sum_{cyc} \frac{x+yz}{x^2+x} \leq 3-3=0$$

Problem 70 *'Claudiu Mandrila'* (Endrit Fejzullahu): Let $a,b,c>0$ such that $abc=1$. Prove that :

$$\frac{a^{10}}{b+c} + \frac{b^{10}}{c+a} + \frac{c^{10}}{a+b} \geq \frac{a^7}{b^7+c^7} + \frac{b^7}{c^7+a^7} + \frac{c^7}{a^7+b^7}$$

Solution (Sayan Mukherjee): Since $abc=1$ so, we have:

$$\sum \frac{a^{10}}{b+c} = \sum \frac{a^7}{b^3c^3(b+c)} \geq \sum \frac{a^7}{\frac{1}{64}(b+c)^6 \cdot (b+c)}$$

So we are only required to prove that: $(b+c)^7 \leq 2^6b^7 + 2^6c^7$
But, from Holder;
$(b^7+c^7)^{\frac{1}{7}}(1+1)^{\frac{6}{7}} \geq b+c$ Hence we are done

Problem 71 *'Hojoo Lee, Crux Mathematicorum'* (Sayan Mukherjee): Let $a,b,c \in \mathbb{R}^+$;Prove that:

$$\frac{2}{abc}(a^3+b^3+c^3) + \frac{9(a+b+c)^2}{a^2+b^2+c^2} \geq 33$$

First Solution (Endrit Fejzullahu): Inequality is equivalent with :

$$\left((a+b+c)(a^2+b^2+c^2)-9abc\right)\left((a-b)^2+(b-c)^2+(c-a)^2\right) \geq 0$$

Second Solution (Popa Alexandru):

$$LHS - RHS = \left(\frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab} - 3\right)\left(2 - \frac{18abc}{(a+b+c)(a^2+b^2+c^2)}\right) \geq 0$$

Problem 72 *'Cezar Lupu'* (Endrit Fejzullahu): Let a, b, c be positive real numbers such that $a+b+c \geq \frac{a}{b}+\frac{b}{c}+\frac{c}{a}$. Show that :

$$\frac{a^3c}{b(c+a)}+\frac{b^3a}{c(a+b)}+\frac{c^3b}{a(b+c)} \geq \frac{3}{2}$$

First Solution (Sayan Mukherjee): Observe that $a+b+c \geq \frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq 3$ (1) from AM-GM. Also from rearrangement inequality:

$$\begin{bmatrix} a; b; c \\ \frac{1}{b}; \frac{1}{c}; \frac{1}{a} \end{bmatrix} \geq \begin{bmatrix} a; b; c \\ \frac{1}{c}; \frac{1}{a}; \frac{1}{b} \end{bmatrix} \implies \frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq \frac{a}{c}+\frac{b}{a}+\frac{c}{b}$$

... (2)
Hence we rewrite LHS of the inequality in the form:

$$\sum_{cyc} \frac{a^2}{b(a+c)\cdot\frac{1}{ac}} = \sum_{cyc} \frac{a^2}{\frac{b}{a}+\frac{b}{c}} \geq \frac{(a+b+c)^2}{\sum_{cyc}\frac{b}{c}+\sum_{cyc}\frac{c}{b}} \geq \frac{(a+b+c)^2}{2\sum_{cyc}\frac{a}{b}} \geq \frac{(a+b+c)}{2} \geq \frac{3}{2}$$

(From (1) (2))
QED.

Second Solution (Popa Alexandru): Holder gives :

$$\sum_{cyc} \frac{a^3c}{b(c+a)} 2(a+b+c)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \geq (a+b+c)^3.$$

Therefore we have :

$$\sum_{cyc} \frac{a^3c}{b(c+a)} \geq \frac{(a+b+c)^3}{2(a+b+c)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)} \geq \frac{3}{2}$$

Third Solution (Popa Alexandru): We have that :

$$\sum_{cyc} \frac{a^3c}{b(c+a)} = \sum_{cyc} \frac{\left(\frac{a}{b}\right)^2}{\frac{1}{b}\left(\frac{1}{c}+\frac{1}{a}\right)}$$

By Cauchy-Schwartz we get :

$$\left(\sum_{cyc}\frac{1}{b}\left(\frac{1}{c}+\frac{1}{a}\right)\right)\left(\sum_{cyc} \frac{\left(\frac{a}{b}\right)^2}{\frac{1}{b}\left(\frac{1}{c}+\frac{1}{a}\right)}\right) \geq \left(\sum_{cyc}\frac{a}{b}\right)^2$$

So:

$$\sum_{cyc} \frac{\left(\frac{a}{b}\right)^2}{\frac{1}{b}\left(\frac{1}{c}+\frac{1}{a}\right)} \geq \frac{\left(\sum_{cyc}\frac{a}{b}\right)\left(\sum_{cyc}\frac{a}{b}\right)}{2\sum_{cyc}\frac{1}{ab}}$$

It'll be enough to prove that :

$$\sum_{cyc}\frac{a}{b} \geq \sum_{cyc}\frac{1}{ab}$$

By contradiction method we assume that $\sum_{cyc} \frac{a}{b} < \sum_{cyc} \frac{1}{ab}$ which is equivalent with $\sum_{cyc} ab^2 < \sum_{cyc} a$. Suming this with the condition $2\sum_{cyc} a \geq 2\sum_{cyc} \frac{a}{b}$, we will have that

$$3\sum_{cyc} a > a\left(b^2 + \frac{2}{b}\right) + b\left(c^2 + \frac{2}{c}\right) + c\left(a^2 + \frac{2}{a}\right)$$

But using AM-GM as $x^2 + \frac{2}{x} = x^2 + \frac{1}{x} + \frac{1}{x} \geq 3\sqrt[3]{x^2 \cdot \frac{1}{x} \cdot \frac{1}{x}} = 3$, we get $3(a+b+c) > 3(a+b+c)$, contradiction .

Problem 73 *'Popa Alexandru'* (Sayan Mukherjee): For $x, y, z > 0$ and $x + y + z = 1$ Find P_{min} if :

$$P = \frac{x^3}{(1-x)^2} + \frac{y^3}{(1-y)^2} + \frac{z^3}{(1-z)^2}$$

Solution (Zhero):
$(3x-1)^2 \geq 0 \iff \frac{x^3}{(1-x)^2} \geq x - \frac{1}{4}$. Adding these up yields that the RHS is $\geq 1 - \frac{3}{4} = \boxed{\frac{1}{4}}$

Problem 74 *'Hlawka'* (Zhero): Let x, y, and z be vectors in $\mathbb{R}^n$. Show that

$$|x+y+z| + |x| + |y| + |z| \geq |x+y| + |y+z| + |z+x|$$

First Solution (Dimitris Charisis): I will solve this inequality for $x, y, z \in \mathbb{C}$.
First observe that $|x+y+z|^2 + |x|^2 + |y|^2 + |z|^2 = |x+y|^2 + |y+z|^2 + |z+x|^2 \boxed{1}$
To prove this just use the well-known identity $|z|^2 = z \cdot \bar{z}$.
Then our inequality is:
$|x+y+z|^2+|x|^2+|y|^2+|z|^2+2(|x+y+z|)(|x|+|y|+|z|)+|xy|+|yz|+|xz| \geq \sum|x+y|^2+2\sum|x+y \parallel y+z|$
From $\boxed{1}$ we only need to prove that:
$(|x+y+z|)(|x|+|y|+|z|) + |xy| + |yz| + |xz| \geq \sum|x+y||y+z|$
But $|(x+y)(y+z)| = |y(x+y+z) + xz| \leq |y(x+y+z)| + |yz|$.
Cyclic we have the result

Second Solution (Popa Alexandru): Use Hlawka's identity :

$$(|a|+|b|+|c|-|b+c|-|c+a|-|a+b|+|a+b+c|) \cdot (|a|+|b|+|c|+|a+b+c|) =$$

$$= (|b|+|c|-|b+c|)\cdot(|a|-|b+c|+|a+b+c|)+(|c|+|a|-|c+a|)\cdot(|b|-|c+a|+|a+b+c|)+(|a|+|b|-|a+b|)\cdot(|c|-|a+b|+|a+b+c|) \geq 0$$

and the conclusion follows .

Problem 75 *'Greek Mathematical Olympiad'* (Dimitris Charisis): If $x, y, z \in \mathbb{R}$ so that $x^2 + 2y^2 + z^2 = \frac{2}{5}a^2, a > 0$,prove that

$$|x - y + z| \leq a$$

Solution (Popa Alexandru): The inequality is equivalent with :

$$a^2 \geq x^2 + y^2 + z^2 - 2yz - 2xy + 2zx$$

Since the condition this rewrites to :

$$\frac{5}{2}x^2 + 5y^2 + \frac{5}{2}z^2 \geq x^2 + y^2 + z^2 - 2yz - 2xy + 2zx$$

which is equivalent with :

$$(x - y)^2 + \left(\frac{x}{\sqrt{2}} + y\sqrt{2}\right)^2 + \left(\frac{z}{\sqrt{2}} + y\sqrt{2}\right)^2 \geq 0$$

Problem 76 *'Vasile Cirtoaje and Mircea Lascu, Junior TST 2003, Romania'* (Hassan Al-Sibyani): Let a, b, c be positive real numbers so that $abc = 1$. Prove that:

$$1 + \frac{3}{a + b + c} \geq \frac{6}{ab + ac + bc}$$

First Solution (Endrit Fejzullahu): Given inequality is equivalent with :

$$ab + bc + ca \geq \frac{6(a + b + c)}{a + b + c + 3} \implies (ab + bc + ca)^2 \geq \frac{36(a + b + c)^2}{(a + b + c + 3)^2}$$

By the AM-GM inequality we know that :

$$(ab + bc + ca)^2 \geq 3abc(a + b + c) = 3(a + b + c)$$

It is enough to prove that

$$3(a + b + c) \geq \frac{36(a + b + c)^2}{(a + b + c + 3)^2} \iff (a + b + c - 3)^2 \geq 0$$

Second Solution (Dimitris Charisis): Setting $a = \frac{1}{x}, b = \frac{1}{y}, c = \frac{1}{z}$ we have to prove that:
$1 + \frac{3}{xy + yz + zx} \geq \frac{6}{x + y + z}$
But $\frac{1}{xy + yz + zx} \geq \frac{3}{(x + y + z)^2}$
So we have to prove that:
$1 + \frac{9}{(x + y + z)^2} \geq \frac{6}{x + y + z}$.
Now setting $x + y + z = k$, the inequality becomes:

$(k-3)^2 \geq 0$ which is true.

Third Solution (Popa Alexandru): We have

$$(ab+bc+ca)^2 \geq 3abc(a+b+c) = 3(a+b+c)$$

Then

$$1+\frac{3}{a+b+c} \geq 1+\frac{9}{(ab+ac+bc)^2} \geq \frac{6}{ab+ac+bc}$$

Problem 77 *'Vasile Pop, Romania 2007 Shortlist'* (Endrit Fejzullahu): Let $a,b,c \in [0,1]$. Show that :

$$\frac{a}{1+bc}+\frac{b}{1+ca}+\frac{c}{1+ab}+abc \leq \frac{5}{2}$$

Solution (Dimitris Charisis):
$bc \leq abc$
So $\dfrac{a}{1+bc} \leq \dfrac{a}{1+abc}$
cyclic we have that $\sum \dfrac{a}{1+bc} \leq \dfrac{a+b+c}{1+abc}$
So our problem reduces to:
$\dfrac{a+b+c}{1+abc}+abc \leq \dfrac{5}{2}.\boxed{1}$. But $a+b+c \leq abc+2$
Because:
$(1-ab)(1-c) \geq 0 \Longleftrightarrow 1-c-ab+abc \geq 0 \Longleftrightarrow 2+abc \geq ab+c+1$.
So we have to prove that $ab+c+1 \geq a+b+c$
But $(a-1)(b-1) \geq 0 \Longleftrightarrow ab-a-b+1 \geq 0 \Longleftrightarrow ab+1 \geq a+b \Longleftrightarrow ab+c+1 \geq a+b+c$
So $\dfrac{a+b+c}{1+abc} \leq \dfrac{abc+2}{abc+1} = 1+\dfrac{1}{abc+1}$
Finally our inequality reduces to:
$\dfrac{1}{abc+1}+abc \leq \dfrac{3}{2}$.
Setting $abc = x \leq 1$ we have:
$\dfrac{1}{x+1}+x \leq \dfrac{3}{2} \Longleftrightarrow 2x^2-x-1 \leq 0 \Longleftrightarrow (x-1)(2x+1) \leq 0$ which is true.

Problem 78 *'Greek 2004'* (Dimitris Charisis): Find the best constant M so that the inequality:

$$x^4+y^4+z^4+xyz(x+y+z) \geq M(xy+yz+zx)^2$$

for all $x,y,z \in \mathbb{R}$.

Solution (Endrit Fejzullahu):
Setting $x=y=z=1$,we see that $M \leq \dfrac{2}{3}$ .We prove that the inequality is true for $M = \dfrac{2}{3}$

$$x^4+y^4+z^4+xyz(x+y+z) \geq \frac{2}{3}(xy+yz+zx)^2$$

After squaring $(xy+yz+zx)^2$ on the RHS ,Inequality is equivalent with :

$$x^4+y^4+z^4+2\left(x^4+y^4+z^4-(xy)^2-(yz)^2-(zx)^2\right)\geq xyz(x+y+z)$$

It is obvious that $x^4+y^4+z^4\geq(xy)^2+(yz)^2+(zx)^2$,so it is enough to prove that :

$$x^4+y^4+z^4\geq xyz(x+y+z)$$

$$x^4+y^4+z^4\geq(xy)^2+(yz)^2+(zx)^2\geq x^2yz+xy^2z+xyz^2=xyz(x+y+z)$$

The last one is true ,setting $xy=a,yz=b,zx=c$,it becomes $a^2+b^2+c^2\geq ab+bc+ca$,which is trivial

Problem 79 *'Dragoi Marius and Bogdan Posa'* (Endrit Fejzullahu): Let $a,b,c>0$ auch that $abc=1$, Prove that :

$$a^2(b^5+c^5)+b^2(a^5+c^5)+c^2(b^5+a^5)\geq 2(a+b+c)$$

Solution (Sayan Mukherjee):
Lemma:
Note that
$a^5+a^5+a^5+b^5+b^5\geq 3a^3b^2$
$a^5+a^5+b^5+b^5+b^5\geq 3a^2b^3$
Summing up;
$a^5+b^5\geq a^2b^2(a+b)$
So $LHS=\sum a^2(b^5+c^5)\geq a^2b^2c^2(c+b)=2(a+b+c)$ from $abc=1$ and our lemma above.

Problem 80 *'Russia 1995'* (Sayan Mukherjee): For positive x,y prove that:

$$\frac{1}{xy}\geq\frac{x}{x^4+y^2}+\frac{y}{x^2+y^4}$$

Solution (FantasyLover):
By AM-GM, we have $x^4+y^2\geq 2x^2y$ and $x^2+y^4\geq 2xy^2$.
Hence, $\dfrac{x}{x^4+y^2}+\dfrac{y}{x^2+y^4}\leq\dfrac{x}{2x^2y}+\dfrac{y}{2xy^2}=\dfrac{1}{xy}$, and we are done. $\blacksquare$

Problem 81 *'Adapted after an IMO problem'* (FantasyLover): Let a,b,c be the side lengths of a triangle with semiperimeter of 1. Prove that

$$1<ab+bc+ca-abc\leq\frac{28}{27}$$

Solution (Sayan Mukherjee): Since $a,b,c\rightarrow$ sides of a triangle; write:
$a=x+y;b=y+z;c=z+x$ so that $x+y+z=1$
The inequality is equivalent to :

$1 < \sum(1-x)(1-y) - \prod(x+y) \le \dfrac{28}{27} \implies 1 < 3 - 2\sum x + xyz = 1 + xyz \le \dfrac{28}{27}$ Since $xyz > 0$ So the left side is proved. And, since $x+y+z=1$ and $xyz \le \left(\dfrac{x+y+z}{3}\right)^3 = \dfrac{1}{27}$ hence the right side is proved

Problem 82 *'Latvia 2002'* (Sayan Mukherjee): Let $a,b,c,d > 0$; $\dfrac{1}{1+a^4} + \dfrac{1}{1+b^4} + \dfrac{1}{1+c^4} + \dfrac{1}{1+a^4} = 1$ Prove that:

$$abcd \ge 3$$

First Solution (Zhero):
Let $w = \dfrac{1}{1+a^4}, x = \dfrac{1}{1+b^4}, y = \dfrac{1}{1+c^4}$, and $z = \dfrac{1}{1+d^4}$.
Then $a = \sqrt[4]{\dfrac{1}{w} - 1}, b = \sqrt[4]{\dfrac{1}{x} - 1}, c = \sqrt[4]{\dfrac{1}{y} - 1}$, and $d = \sqrt[4]{\dfrac{1}{z} - 1}$.
Let $f(n) = -\ln(\sqrt[4]{\dfrac{1}{n} - 1})$. It is easy to verify that f has exactly one inflection point. Since we see seek to minimize $f(w) + f(x) + f(y) + f(z)$, by the inflection point theorem, it suffices to minimize it in the case in which $w = x = y$. In other words, in our original inequality, it suffices to minimize this in the case that $a = b = c$.
Let $p = a^4, q = b^4, r = c^4$, and $s = d^4$. Then we want to show that $pqrs \ge 81$ when $\sum_{cyc} \dfrac{1}{1+p} = 1$. But we only need to check this in the case in which $p = q = r$, that is, when $s = \dfrac{3}{p-2}$. In other words, we want to show that $\dfrac{3p^3}{p-2} \ge 81 \iff 3p^3 \ge 81p - 162 \iff 3(x-3)^2(x+6) \ge 0$, as desired.
Second Solution (Sayan Mukherjee):
$\dfrac{d^4}{1+d^4} = 1 - \dfrac{1}{1+d^4} = \sum_{a,b,c} \dfrac{1}{1+a^4} \ge \dfrac{3}{\sqrt[3]{\prod_{a,b,c}(1+a^4)}}$(AM-GM)
Similarly we get three ther relations and multiplying we get:

$$\frac{a^4b^4c^4d^4}{\prod_{a,b,c,d}(1+a^4)} \ge \frac{3^4}{\prod_{a,b,c,d}(1+a^4)} \implies abcd \ge 3$$

Problem 83 *'Vasile Cartoaje and Mircea Lascu'* (Zhero): Let a, b, c, x, y, and z be positive real numbers such that $a + x \ge b + y \ge c + z and a + b + c = x + y + z$. Show that

$$ay + bx \ge ac + xz.$$

Solution (mathinequs):
$a(y-c) + x(b-z) = a(a-x) + (a+x)(b-z)$
Now i used :
$a(a-x) = \dfrac{(a-x)^2 + a^2 - x^2}{2}$

So i got that :

$a(a-x)+(a+x)(b-z)=\dfrac{(a-x)^2+(a+x)(a-x+2(b-z))}{2}$

So i have to prove that : $(a+x)(a-x+2(b-z))\geq 0$

but $a+x$ is greater or equal than 0 and $a-x+2(b-z)=b+y-c-z$ which is greater or equal than 0.

Problem 84 *'Nicolae Paun'* (mathinequs): Let a,b,c,x,y,z reals with $a+b+c=x^2+y^2+z^2=1$. To prove:

$$a(x+b)+b(y+c)+c(z+a)<1$$

Solution (Mharchi Abdelmalek): Using AM-GM inequality we have: $a^2+x^2+b^2+y^2+c^2+z^2\geq 2ax+2by+2cz$

Hence we have :

$1=\dfrac{(a+b+c)^2+x^2+y^2+z^2}{2}=\dfrac{1}{2}\left(\sum_{cyc}(a^2+x^2)\right)+ab+bc+ca\geq ax+by+cz+ab+bc+ca=$

$=a(x+b)+b(y+c)+c(z+a)$

Problem 85 *'Asian Pacific Mathematics Olympiad'* (Mharchi Abdelmalek): Let $x,y,z,$ be a positive real numbers . Prove that:

$$\left(\frac{x}{y}+1\right)\left(\frac{y}{z}+1\right)\left(\frac{z}{x}+1\right)\geq 2+\frac{2(x+y+z)}{\sqrt[3]{xyz}}$$

Solution (mathinequs):

Bashing the left hand side we need to prove :

$\sum\dfrac{x}{y}+\sum\dfrac{y}{x}\geq\dfrac{2(x+y+z)}{\sqrt[3]{xyz}}$

But this is AM-GM as :

$2\dfrac{x}{y}+\dfrac{y}{z}\geq\dfrac{3x}{\sqrt[3]{xyz}}$

Done.

Problem 86 *'mateforum.ro'* (mathinequs): Let $x,y,z>0$ such that $xy+yz+zx=1$. Show:

$$\frac{1}{x+x^3}+\frac{1}{y+y^3}+\frac{1}{z+z^3}\geq\frac{9\sqrt{3}}{4}$$

Solution (Apartim De):

The condition $xy+yz+zx=1$ accomodates the substitution

$x=\cot A, y=\cot B,\ z=\cot C,$ where A,B,C are the angles of ΔABC.

We obtain the equivalent inequality

$\otimes\sum_{cyc}\dfrac{1}{\cot A(1+\cot^2 A)}\geqslant\dfrac{9\sqrt{3}}{4}$

$\Leftrightarrow \sum_{cyc} \tan A \sin^2 A \geqslant \frac{9\sqrt{3}}{4}$
Let $f(x) = \tan x \sin^2 x$
• $f'(x) = 2\sin^2 x + \tan^2 x > 0, \forall x \Leftrightarrow f(x)$ is increasing
$f''(x) = 2\sin x \left[2\cos x + \frac{1}{\cos^3 x}\right] \begin{cases} > 0, \forall x \in (0, \pi/2) \\ < 0, \forall x \in (\pi/2, \pi) \\ > 0, \forall x \in (\pi, 3\pi/2) \end{cases}$
If the ΔABC be acute , by Jensen,
$\sum_{cyc} f(A) \geqslant 3f\left(\frac{\pi}{3}\right) = \frac{9\sqrt{3}}{4}$
If the ΔABC be obtuse-angled(at A) ,
$f(A) + f(B) + f(C) = f\left(A + \frac{\pi}{2}\right) + f(B-k) + f\left(C - \left(\frac{\pi}{2} - k\right)\right)$
$\geqslant 3f\left(\frac{\left(A + \frac{\pi}{2}\right) + (B-k) + \left(C - \left(\frac{\pi}{2} - k\right)\right)}{3}\right) = 3f\left(\frac{\pi}{3}\right)$
$f$or some suitable$k \leqslant \min\{B, C\}$

Problem 87 '—' (Apartim De): a, b, c are sides of a triangle.Prove that

$$\frac{a}{\sqrt{2b^2+2c^2-a^2}} + \frac{b}{\sqrt{2c^2+2a^2-b^2}} + \frac{c}{\sqrt{2a^2+2b^2-c^2}} \geqslant \sqrt{3}$$

First Solution (Endrit Fejzullahu): According to the Holder's inequality we have :

$$\left(\sum_{cyc} \frac{a}{\sqrt{2b^2+2c^2-a^2}}\right)^2 \left(\sum_{cyc} a(2b^2+2c^2-a^2)\right) \geq (a+b+c)^3$$

We only need to prove that

$$(a+b+c)^3 \geq 3\sum_{cyc} a(2b^2+2c^2-a^2)$$

It can be rewritten :

$$3\left(abc - (a+c-b)(b+c-a)(a+b-c)\right) + 2\left(a^3+b^3+c^3-3abc\right) \geq 0$$

By Schur $abc \geq (a+c-b)(b+c-a)(a+b-c)$ and by AM-GM $a^3+b^3+c^3 \geq 3abc$,so we're done !

Second Solution (geniusbliss):

$$\sum_{cyclic} \frac{a}{\sqrt{2b^2+2c^2-a^2}} + \sum_{cyclic} \frac{a}{\sqrt{2b^2+2c^2-a^2}} + \sum_{cyclic} \frac{3\sqrt{3}a(2b^2+2c^2-a^2)}{(a+b+c)^3} \geq 3\sum_{cyclic} \frac{\sqrt{3}a}{a+b+c} = 3\sqrt{3}$$

and finish off the problem

Problem 88 *'Pavel Novotn, Slovakia'* (Endrit Fejzullahu): Let a,b,c,d be positive real numbers such that $abcd = 1$ and $a+b+c+d > \dfrac{a}{b}+\dfrac{b}{c}+\dfrac{c}{d}+\dfrac{d}{a}$. Prove that

$$a+b+c+d < \frac{b}{a}+\frac{c}{b}+\frac{d}{c}+\frac{a}{d}$$

First Solution (mathinequs): It's AM-GM ! $\frac{a}{d}+2\frac{a}{b}+\frac{b}{c} \geq 4a$ Suming the other and with condition : $3(a+b+c+d)+\frac{b}{a}+\frac{c}{b}+\frac{d}{c}+\frac{a}{d} > 3\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}\right)+\frac{b}{a}+\frac{c}{b}+\frac{d}{c}+\frac{a}{d} \geq 4(a+b+c+d)$

Second Solution (socrates): The second condition implies that

$$a+b+c+d > \frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a} \geq \frac{a^2}{ab}+\frac{b^2}{bc}+\frac{c^2}{cd}+\frac{d^2}{da} \geq \frac{(a+b+c+d)^2}{ab+bc+cd+da}$$

so $ab+bc+cd+da > a+b+c+d$. Now, it is

$$\frac{b}{a}+\frac{c}{b}+\frac{d}{c}+\frac{a}{d} = \frac{(bc)^2}{abc^2}+\frac{(cd)^2}{bcd^2}+\frac{(da)^2}{cda^2}+\frac{(ab)^2}{dab^2} = \frac{(bc)^2}{\frac{c}{d}}+\frac{(cd)^2}{\frac{d}{a}}+\frac{(da)^2}{\frac{a}{b}}+\frac{(ab)^2}{\frac{b}{c}} >$$

$$\frac{(bc+cd+da+ab)^2}{\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}} > \frac{(a+b+c+d)^2}{a+b+c+d} = a+b+c+d$$

completing the proof.

Problem 89 *'—'* (geniusbliss): Let $a_1, a_2, a_3 ..., a_n$ and $b_1, b_2, b_3, ..., b_n$ be real numbers such that - $a_1 \geq \dfrac{a_1+a_2}{2} \geq \dfrac{a_1+a_2+a_3}{3} \geq .... \geq \dfrac{a_1+a_2+a_3+...+a_n}{n}$, $b_1 \geq \dfrac{b_1+b_2}{2} \geq \dfrac{b_1+b_2+b_3}{3} \geq .... \geq \dfrac{b_1+b_2+b_3+...+b_n}{n}$, then prove that -

$$a_1b_1+a_2b_2+a_3b_3+...+a_nb_n \geq \frac{1}{n}\cdot(a_1+a_2+a_3+...+a_n)\cdot(b_1+b_2+b_3+...+b_n)$$

Solution (geniusbliss): For each $k1, 2, ..., n$,to we denote $S_k = a_1+a_2+a_3+...+a_k$ and $b_{n+1} = 0$
Then by Abel's Formulae, we have
$\sum_{i=1}^{n} a_ib_i = \sum_{i=1}^{n}(b_i - b_{i+1})S_i = \sum_{i=1}^{n} i(b_i - b_{i+1})\frac{S_i}{i}$
According to Abel's Formulae,
$\sum_{i=1}^{n} a_ib_i = (S_1 - \frac{S_2}{2})(b_1 - b_2) + (\frac{S_2}{2} - \frac{S_3}{3})(b_1 + b_2 - 2b_3) + ... + (\frac{S_{n-1}}{n-1} - \frac{S_n}{n})(\sum_{i=1}^{n-1} -(n-1)b_n) + \frac{1}{n}(\sum_{i=1}^{n} a_1)(\sum_{i=1}^{n} b_1)$
By Hypothesis, we have $\frac{S_1}{1} \geq \frac{S_2}{2} \geq ..... \geq \frac{S_n}{n}$ so its enough to prove that
$\sum_{i=1}^{k} b_i \geq kb_{k+1}$
This one comes directly from the hypothesis
$\frac{1}{k}\sum_{i=1}^{k} b_i \geq \sum_{i=1}^{k+1} b_i$

Problem 90 *'mateforum.ro'* (mathinequs): Let a, b, c positives with $\sum \frac{1}{1+a^2} = 2$. Show :

$$abc(a+b+c-abc) \leq \frac{5}{8}$$

Solution (socrates): Multiplying out we get $2a^2b^2c^2 + a^2b^2 + b^2c^2 + c^2a^2 = 1$.
Moreover,

$$abc(a+b+c-abc) = \frac{1}{2}\left(2abc(a+b+c) - 2a^2b^2c^2\right) = \frac{1}{2}\left(2abc(a+b+c) + a^2b^2 + b^2c^2 + c^2a^2 - 1\right) = \frac{1}{2}\left((ab+bc+ca)^2 - 1\right)$$

so it is enough to prove that $ab + bc + ca \leq \frac{3}{2}$
Now put $a^2 = \frac{x}{y+z}, b^2 = \frac{y}{x+z}, c^2 = \frac{z}{x+y},\ x, y, z > 0$. Then
$ab = \sqrt{\frac{xy}{(y+z)(x+z)}} \leq \frac{1}{2}\left(\frac{x}{x+z} + \frac{y}{y+z}\right)$ and similarly
$bc = \sqrt{\frac{yz}{(x+z)(x+y)}} \leq \frac{1}{2}\left(\frac{z}{x+z} + \frac{y}{x+y}\right)$
$ac = \sqrt{\frac{xz}{(y+z)(x+y)}} \leq \frac{1}{2}\left(\frac{x}{x+y} + \frac{z}{y+z}\right)$
Adding these, we are done.

Problem 91 *'Komal'* (Hassan Al-Sibyani): For arbitrary real numbers a, b, c. Prove that:

$$\sqrt{a^2 + (1-b)^2} + \sqrt{b^2 + (1-c)^2} + \sqrt{c^2 + (1-a)^2} \geq \frac{3\sqrt{2}}{2}$$

First Solution (Dimitris Charisis): By Minkowski's inequality we have:
$LHS \geq \sqrt{(a+b+c)^2 + (a+b+c-3)^2}$.
Setting $a + b + c = x$ our inequality becomes:

$$\sqrt{x^2 + (x-3)^2} \geq \frac{3\sqrt{2}}{2} \Longleftrightarrow 2x^2 - 6x + \frac{9}{2} \geq 0 \Longleftrightarrow x^2 - 3x + \frac{9}{4} \geq 0 \Longleftrightarrow \left(x - \frac{3}{2}\right)^2 \geq 0$$

,which is true....

Second Solution (Mharchi Abdelmalek): we have the flowing well-know inequality for all real numbers a, b :
$\sqrt{a^2 + (1-b)^2} \geq \frac{|a+1-b|}{\sqrt{2}}$
Simarly for the other numbers.Hence we have

$$\text{LHS} \geq \frac{|a-b+1| + |b-c+1| + |c-a+1|}{\sqrt{2}} \geq \frac{|a-b+b-c+c-a+3|}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$$

Problem 92 *'Asian Pacific Mathematics Olympiad 1996'* (Dimitris Charisis): For a, b, c sides of a triangle prove that:

$$\sqrt{a+b-c}+\sqrt{a-b+c}+\sqrt{-a+b+c} \leq \sqrt{a}+\sqrt{b}+\sqrt{c}$$

First Solution (geniusbliss): substitute $a = x+y, b = y+z, c = z+x$ we have to prove then,
$\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x} \geq \sqrt{2x}+\sqrt{2y}+\sqrt{2z}$ which is true because squaring both sides,
we have to prove $\sum_{cyc} \sqrt{(x+y)(y+z)} \geq 2\sqrt{xy}+2\sqrt{yz}+2\sqrt{zx}$ which is true by Cauchy - $(x+y)(y+z) \geq (\sqrt{xy}+\sqrt{yz})^2$ and
similarly for others ,equality holds for $x = y = z$

Second Solution (Redwane Khyaoui): First of all , we set $a = \dfrac{x+y}{2}$ and $b = \frac{y+z}{2}$ and $c = \frac{z+x}{2}$
the inequality become into : $\sqrt{2}(\sqrt{x}+\sqrt{y}+\sqrt{z}) \leq \sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}$
then we use the well-known inequality : $\dfrac{1}{2}(\sqrt{x}+\sqrt{y})^2 \leq x+y$ which means $\sqrt{2(x+y)} \geq \sqrt{x}+\sqrt{y}$
and then summing the three inequalities , gives inequality desired .

Problem 93 *'geniusbliss'* (geniusbliss): Prove that for x, y, z positive reals such as $xz \geq y^2, xy \geq z^2$ the follwing inequality holds-

$$(x^2-yz)^2 \geq \frac{27}{8}(xz-y^2)(xy-z^2)$$

Solution (geniusbliss): this obviously implies the LHS is also positive.
make this substitution - $a = \dfrac{x}{y}, b = \dfrac{y}{z}, c = \dfrac{z}{x}$,
the inequality gets transformed into
$8(a-c)^2 \geq 27(a-b)(b-c)$ or
$2((a-c)^2)^{\frac{1}{3}} \geq 3((a-b)(b-c))^{\frac{1}{3}}$ we have $a \geq b \geq c$,
multiply both sides by $(a-c)^{\frac{1}{3}}$, and we get,
$2(a-c) \geq 3((a-b)(b-c)(a-c))^{\frac{1}{3}}$ but this is just AM-GM if we write the LHS as $2(a-c) = a-b+b-c+a-c$
so we are done and the inequality is true with equality holding when $a = b = c$

Problem 94 *'Kvant 1988'* (Mharchi Abdelmalek): Let $a, b, c \geq 0$ such that $a^4+b^4+c^4 \leq 2(a^2b^2+b^2c^2+c^2a^2)$
Prove that :

$$a^2+b^2+c^2 \leq 2(ab+bc+ca)$$

Solution (Hassan Al-Sibyani): The condition

$$\sum a^4 \leq 2\sum a^2b^2$$

is equivalent to

$$(a+b+c)(a+b-c)(b+c-a) \geq 0$$

In any of the cases $a = b + c, b = c + a, c = a + b$, the inequality

$$\sum a^2 \leq 2 \sum ab$$

is clear. So, suppose $a \neq b+c, b \neq c+a, c \neq a+b$. Because at most one of the numbers $b+c-a, c+a-b, a+b-c$ is negative and their products is non-negative, all of them are positive. Thus, we may assume that:

$$a^2 < ab + ac, b^2 < bc + ba, c^2 < ca + cb$$

and the conclusion follows.

Problem 95 *'Junior Balkan Mathematical Olympiad 2002 Shortlist'* (Hassan Al-Sibyani): Let a, b, c be positive real numbers. Prove that

$$\frac{a^3}{b^2} + \frac{b^3}{c^2} + \frac{c^3}{a^2} \geq \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a}$$

First Solution (Redwane Khyaoui): If $a \geq b \geq c$ then we put $a = tb$ and $b = t'c$ so $a = tt'c$ with $t, t' \geq 1$
th inequality become into $t^5t'^3 + t'^5.t^2 + 1 \geq t^4t'^3 + t'^4.t^2 + tt'$
and we consider the fonction $f(t) = t^5t'^3 + t'^5.t^2 + 1 - t^4t'^3 + t'^4.t^2 + tt'$
so $f'(t) = 5t'^3t^4 - 4t'^3t^3 + 2t(t'^5 - t'^4) - tt' + 1$
then , since we know that $t, t' \geq 1$ so $f'(t) \geq 0$ which means f is increasing function ,
$t \geq 1 \rightarrow f(t) \geq f(1) = t'^5 - t'^4 - t' + 1$
setting again $g(t') = t'^5 - t'^4 - t' + 1$ so $g'(t) = 5t'^4 - 4t'^3 - 1 \geq 0$ because $t' \geq 1$
whic means that g is increasing fonction , so $t' \geq 1 \rightarrow g(t) \geq g(1) = 0$
and finally $t \geq 1 \rightarrow f(t) \geq f(1) = t'^5 - t'^4 - t' + 1 \geq 0$

Second Solution (Mharchi Abdelmalek):

$$\text{LHS}-\text{RHS} \geq 0 \Leftrightarrow \sum_{cyc}\left(\frac{a^3}{b^2} - \frac{2a^2}{b} + a\right) + \sum_{cyc}\left(\frac{a^2}{b} - 2a + b\right) \geq 0 \Leftrightarrow \sum_{cyc}\left(\frac{\sqrt{a^3}}{b} - \sqrt{a}\right)^2 + \sum_{cyc}\left(\frac{(a-b)^2}{b}\right) \geq 0$$

Which is Obviously true!

Problem 96 *'Hojoo Lee'* (Redwane Khyaoui): If $a + b + c = 1$ then prouve that

$$\frac{a}{a+bc} + \frac{b}{b+ac} + \frac{\sqrt{abc}}{c+ab} \leq 1 + \frac{3\sqrt{3}}{4}$$

Solution (Endrit Fejzullahu): Let $x = \sqrt{\dfrac{ab}{c}}, y = \sqrt{\dfrac{bc}{a}}, z = \sqrt{\dfrac{ca}{b}}$.We see that $xy + yz + zx = 1$,and $a = zx, b = xy$ and $c = yz$
Then we can write : (A,B,C are angles of an acute angled triangle)

$$x = \tan\frac{A}{2}\tan\frac{B}{2}, y = \tan\frac{B}{2}\tan\frac{C}{2}, z = \tan\frac{C}{2}\tan\frac{A}{2}$$

Then $a = \tan^2 \dfrac{A}{2} \tan \dfrac{B}{2} \tan \dfrac{C}{2}$, and so on ...
After making the substitutions ,and using trigonometric transformations formulas ,the given inequality becomes :

$$\frac{1}{1+\tan^2 \frac{C}{2}} + \frac{1}{1+\tan^2 \frac{B}{2}} + \frac{\tan \frac{A}{2}}{1+\tan^2 \frac{A}{2}} \leq + \frac{3\sqrt{3}}{4} \iff \cos C + \cos B + \sin A \leq \frac{3\sqrt{3}}{2}$$

$$\cos C + \cos B + \sin C \leq \sin \frac{C}{2} + \sin C < \frac{3\sqrt{3}}{2} \iff 12x^2 < 27 + 16x^4$$

where $x = \sin \dfrac{C}{2}$

Problem 97 *'—'* (Endrit Fejzullahu): If $a_1, a_2, ..., a_n$ are positive real numbers such that $a_1 + a_2 + ... + a_n = 1$.Prove that

$$\sum_{i=1}^{n} \frac{a_i}{2 - a_i} \geq \frac{n}{2n-1}$$

Solution (Mharchi Abdelmalek): Using The Cauchy Shwarz inequality
we have

$$\sum_{i=1}^{n} \frac{a_i}{2-a_i} = \sum_{i=1}^{n} \frac{a_i^2}{2a_i - a_i^2} \geq \frac{(a_1 + a_2 ... + a_n)^2}{2(a_1 + a_2 ... + a_n) - \sum a_i^2} = \frac{1}{2 - \sum a_i^2} \geq \frac{1}{2 - \frac{1}{n}} = \frac{n}{2n-1}$$

Problem 98 *'MOSP 2001'* (Mharchi Abdelmalek): Prove that if $a, b, c > 0$ have product 1 then :

$$(a+b)(b+c)(c+a) \geq 4(a+b+c-1)$$

Solution (Endrit Fejzullahu): Since $abc = 1$
Then

$$(a+b)(b+c)(c+a) = (a+b+c)(ab+bc+ca) - 1 \geq 4(a+b+c-1) \iff ab+bc+ca + \frac{3}{a+b+c} \geq 4$$

By the Am-Gm inequality

$$3\left(\frac{ab+bc+ca}{3}\right) + \frac{3}{a+b+c} \geq 4\sqrt[4]{\frac{3(ab+bc+ca)^3}{27(a+b+c)}}$$

It is enough to prove that
$3(ab+bc+ca)^3 \geq 27(a+b+c) or (ab+bc+ca)^3 \geq 9(a+b+c)$
By AM-GM ,we know that $(ab+bc+ca)^2 \geq 3abc(a+b+c) = 3(a+b+c)$, so it suffices to prove $ab+bc+ca \geq 3$,which is obvious .

Problem 99 *'—'* (Endrit Fejzullahu): Let m, n be natural numbers .Prove that

$$\sin^{2m} x \cdot \cos^{2n} y \leq \frac{m^n n^n}{(m+n)^{m+n}}$$

Solution (socrates): Use AM-GM (or weighted AM-GM) to get

$$1 = \sin^2 x + \cos^2 x = m \cdot \frac{\sin^2 x}{m} + n \cdot \frac{\cos^2 x}{n} =$$
$$\underbrace{\frac{\sin^2 x}{m} + \frac{\sin^2 x}{m} + ... + \frac{\sin^2 x}{m}}_{m\ times} + \underbrace{\frac{\cos^2 x}{n} + \frac{\cos^2 x}{n} + ... \frac{\cos^2 x}{n}}_{n\ times} \geq$$
$$\geq (m+n)\sqrt[m+n]{(\tfrac{\sin^2 x}{m})^m (\tfrac{\cos^2 x}{n})^n}$$

etc...

Problem 100 *'Mircea Lascu'* (Mircea Lascu): Let $a, b, c > 0$. Prove that

$$\frac{c}{a} + \frac{a}{b+c} + \frac{b}{c} \geq 2$$

Solution (Vo Quoc Ba Can): Write the inequality as

$$\frac{c}{a} + \frac{a}{b+c} + \frac{b+c}{c} \geq 3$$

In this form, we can see immediately that it is a direct consequence of the AM-GM Inequality.