ANALYSIS AGAINST NUMBER THEORY?

"Olympiad problems can be solved without the aid of analysis or linear algebra" is a sentence always heard when speaking about the elementary problems given in contests. This is true, but the true nature and essence of some of these problems is in analysis and this is the reason for which such type of problems are always the highlight of a contest. Their elementary solutions are very tricky and sometimes extremely difficult, while using analysis they can be solved quickly. Well, of course, "quickly" only if you see the sequence that hides after each problem. Practically, our aim is to exhibit convergent sequences formed by integer numbers. These sequences must become constant and from here the problem is much easier. The difficulty is in finding those sequences. Sometimes, this is easy, but most of the time this is a very difficult task. We will develop our skills in "hunting" these sequences by solving first some easy problems (anyway, "easy" is a relative concept: try to solve them elementary and you will see if they really are easy) and after that we will attack the chestnuts.

As usual, we begin with a classic beautiful problem, which has lots of applications and extensions.

Example 1. Let $f, g \in Z[X]$ be two non-constant polynomials such that $f(n)|g(n)$ for an infinite natural numbers n. Prove that f divides g in $Q[X]$.

Solution. Indeed, we need to look at the remainder of g when divided with f in $Q[X]$! Let us write $g = fh + r$, were h, r are polynomials from $Q[X]$ and $\deg r < \deg f$. Now, multiplying by the common denominator of all coefficients of polynomials h, r, the hypothesis becomes: there exists two infinite sequences $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$ of integer numbers and a positive integer N such that $b_n = N\dfrac{r(a_n)}{f(a_n)}$ (we could have some problems with the roots of f, but they are in finite number and the sequence $(a_n)_{n\geq 1}$ tends to infinity, so from a certain point, a_n is not a root of f). Since $\deg r < \deg f$, it follows that $\dfrac{r(a_n)}{f(a_n)} \to 0$, thus $(b_n)_{n\geq 1}$ is a sequence of integer numbers that tends to 0. This implies that from a certain point, all the terms of these sequence are 0. Well, this is the same as $r(a_n) = 0$ from a certain point n_0, which is practically the same thing with $r = 0$ (don't forget that any non-zero polynomial has only a finite number of roots!). But in this moment the problem is solved.

The next problem we are going to discuss is a particular case of a much more general and classical result: if f is a polynomial with integer coefficients, $k > 1$ is a natural number and $\sqrt[k]{f(n)} \in Q$ for all natural numbers n, then there exists a polynomial $g \in Q[X]$ such that $f(x) = g^k(x)$. We won't discuss here this general result (the reader will find a proof in the chapter about arithmetic properties of polynomials).

Example 2. Let $a \neq 0, b, c$ be integers such that for any natural number n, the number $an^2 + bn + c$ is a perfect square. Prove that there exist $x, y \in Z$ such that $a = x^2$, $b = 2xy$, $c = y^2$.

Solution. Let us begin by writing $an^2 + bn + c = x_n^2$ for a certain sequence of nonnegative integers $(x_n)_{n\geq 1}$. We could expect that $x_n - n\sqrt{a}$ converges. And yes, it converges, but it's not a sequence of integers, so the convergence is useless. In fact, it's not that useless, but we need another sequence. The easiest way is to work with $(x_{n+1} - x_n)_{n\geq 1}$, since this sequence certainly converges to $\sqrt{a}$ (the reader has already noticed why it wasn't useless to find that $x_n - n\sqrt{a}$ is convergent; we used this to establish the convergence of $(x_{n+1} - x_n)_{n\geq 1}$). This time, the sequence is formed by integer numbers, so it is constant from a certain point. Thus, we can find a number M such that if $n \geq M$ then $x_{n+1} = x_n + \sqrt{a}$. Thus, a must be a perfect square, let us say $a = x^2$. A simple induction shows that $x_n = x_M + (n - M)x$ and so $(x_M - Mx + nx)^2 = x^2n^2 + bn + c$ for all $n \geq M$. A simple identification of coefficients finishes the solution, since we can take $y = x_M - Mx$.

The following problem is based on the same idea, but it really doesn't seem to be related with mathematical analysis. In fact, as we will see, it is closely related to the concept of convergence.

Example 3. Let $a, b, c > 1$ be positive integers such that for any positive integer n there exists a positive integer k such that $a^k + b^k = 2c^n$. Prove that $a = b$.

Laurentiu Panaitopol

Solution. What does the problem say in fact? That we can find a sequence of positive integers $(x_n)_{n\geq 1}$ such that $a^{x_n} + b^{x_n} = 2c^n$. What could be the convergent sequence here? We see that $(x_n)_{n\geq 1}$ is appreciatively kn for a certain constant k. Thus, we could expect that the sequence $(x_{n+1} - x_n)_{n\geq 1}$ converges. Let us see if this is true or not. From where could we find $x_{n+1} - x_n$? Certainly, by writing that $a^{x_{n+1}} + b^{x_{n+1}} = 2c^{n+1}$ and after that considering the value $\dfrac{a^{x_{n+1}} + b^{x_{n+1}}}{a^{x_n} + b^{x_n}} = c$. Now, let us suppose that $a > b$ and let us write $\dfrac{a^{x_{n+1}} + b^{x_{n+1}}}{a^{x_n} + b^{x_n}} = c$ in the form

$$a^{x_{n+1} - x_n} \frac{1 + \left(\dfrac{b}{a}\right)^{x_{n+1}}}{1 + \left(\dfrac{b}{a}\right)^{x_n}} = c,$$

from where it is easy to see that $a^{x_{n+1} - x_n}$ converges to c. Why is it so easy? It would be easy if we could show that $x_n \to \infty$. Fortunately, this is immediate, since $2a^{x_n} > 2c^n \Rightarrow x_n > n \log_a c$. So, we found that $a^{x_{n+1} - x_n}$ converges. Being a sequence of integer numbers, it must become constant, so there exist M such that for all $n \geq M$ we have

$a^{x_{n+1}-x_n} = c$. This means that for all $n \geq M$ we also have

$$\frac{1+\left(\frac{b}{a}\right)^{x_{n+1}}}{1+\left(\frac{b}{a}\right)^{x_n}} = 1.$$

But this is impossible, since $a > b$. Thus, our assumption was wrong and we must have $a \leq b$. Due to symmetry in a and b, we conclude that $a = b$.

Another easy example is the following problem, in which finding the right convergent sequence of integers in not difficult at all. But, attention must be paid to details!

Example 4. Let $a_1, a_2, \ldots, a_k$ be positive real numbers such that at least one of them is not an integer. Prove that there exits infinitely many natural numbers n such that n and $[a_1 n] + [a_2 n] + \cdots + [a_k n]$ are relatively prime.

Gabriel Dospinescu, Arhimede Magazine

Solution. Of course, the solution of such a problem is better to be indirect. So, let us assume that there exists a number M such that for all $n \geq M$ the numbers n and $[a_1 n] + [a_2 n] + \cdots + [a_k n]$ are not relatively prime. Now, what are the most efficient numbers n to be used? Yes, they are the prime numbers, since if n is prime and it is not relatively prime with , $[a_1 n] + [a_2 n] + \cdots + [a_k n]$, then it must divide $[a_1 n] + [a_2 n] + \cdots + [a_k n]$. This suggests us to consider the sequence of prime numbers $(p_n)_{n\geq 1}$. Since this sequence is infinite, there is a number N such that if $n \geq N$ then $p_n \geq M$. According to our assumption, this implies that for all $n \geq N$ there exist a natural number x_n such that $[a_1 p_n] + [a_2 p_n] + \cdots + [a_k p_n] = x_n p_n$. And now, you have already guessed what is the convergent sequence! Yes, it is $(x_n)_{n\geq N}$. This is obvious, since $\frac{[a_1 p_n] + [a_2 p_n] + \cdots + [a_k p_n]}{p_n}$ tends to $n \geq N a_1 + a_2 + \cdots + a_k$.

Thus, we can find a number P such that for $x_n = a_1 + a_2 + \cdots + a_k$ for all $n \geq P$. But this is the same as $\{a_1 p_n\} + \{a_2 p_n\} + \cdots + \{a_k p_n\} = 0$. Of course, this says that $a_i p_n \in Z$ for all $i = \overline{1,k}$ and $n \geq P$. Well, the conclusion is immediate: $a_i \in Z$ for all $i = \overline{1,k}$, which contradicts the hypothesis. Consequently, we were wrong again and the problem statement is right!

Step by step, we start to have some experience in "guessing" the sequences. Thus, it's time to solve some more difficult problems. The next problem we are going to discuss may seem obvious after reading the solution. In fact, it's just that type of problem whose solution is very short, but very hard to find.

Example 5. Let $a, b \in Z$ such that for all natural numbers n the number $a \cdot 2^n + b$ is a perfect square. Prove that $a = 0$.

Poland TST

Solution. Again, we argue by contradiction. Suppose that $a \neq 0$. Then, of course, $a > 0$, otherwise for large values of n the number $a \cdot 2^n + b$ is negative. According to the hypothesis, there exists a sequence of positive integers $(x_n)_{n \geq 1}$ such that for all natural numbers n, $x_n = \sqrt{a \cdot 2^n + b}$. Then, a direct computation shows that $\lim_{n \to \infty} (2x_n - x_{n+2}) = 0$. This implies the existence of a natural number N such that for all $n \geq P$ we have $2x_n = x_{n+2}$. But $2x_n = x_{n+2}$ is equivalent with $b = 0$. Then, a and $2a$ are both perfect squares, which is impossible for $a \neq 0$. This shows, as usually, that our assumption was wrong and indeed $a = 0$.

A classical result of Schur states that for any non-constant polynomial f with integer coefficients, the set of prime numbers dividing at least one of the numbers $f(1), f(2), f(3), \ldots$ is infinite. The following problem is a generalization of this result.

Example 6. Suppose that f is a polynomial with integer coefficients and (a_n) is a strictly increasing sequence of natural numbers such that

$a_n \leq f(n)$ for all n. Then the set of prime numbers dividing at least one term of the sequence is infinite.

Solution. The idea is very nice: for any finite set of prime numbers $p_1, p_2, \ldots, p_r$ and any $k > 0$, we have

$$\sum_{\alpha_1, \alpha_2, \ldots, \alpha_N \in Z_+} \frac{1}{p_1^{k\alpha_1} \ldots p_N^{k\alpha_N}} < \infty.$$

Indeed, it suffices to remark that we have actually

$$\sum_{\alpha_1, \alpha_2, \ldots, \alpha_N \in Z_+} \frac{1}{p_1^{k\alpha_1} \ldots p_N^{k\alpha_N}} \prod_{j=1}^{N} \sum_{i \geq 0} \frac{1}{p_j^{ki}} = \prod_{j=1}^{n} \frac{p_j^k}{p_j^k - 1}.$$

On the other hand, by taking $k = \dfrac{1}{2\deg(f)}$ we clearly have

$$\sum_{n \geq 1} \frac{1}{(f(n))^k} = \infty.$$

Thus, if the conclusion of the problem is not true, we can find $p_1, p_2, \ldots, p_r$ such that any term of the sequence is of the form $p_1^{k\alpha_1} \ldots p_N^{k\alpha_N}$ and thus

$$\sum_{n \geq 1} \frac{1}{a_n^k} \leq \sum_{\alpha_1, \alpha_2, \ldots, \alpha_N \in Z_+} \frac{1}{p_1^{k\alpha_1} \ldots p_N^{k\alpha_N}} < \infty.$$

On the other hand, we also have

$$\sum_{n \geq 1} \frac{1}{a_n^k} \geq \sum_{n \geq 1} \frac{1}{(f(n))^k} = \infty,$$

which is clearly impossible.

The same idea is employed in the following problem.

Example 7. Let $a, b \geq 2$ be natural numbers. Prove that there is a multiple of a which contains all digits $0, 1, \ldots, b-1$ when written in base b.

Adapted after a Putnam problem

Solution. Let's suppose the contrary. Then any multiple of a misses at least a digit when written in base b. Since the sum of inverses of all multiples of a diverges (because $1+\frac{1}{2}+\frac{1}{3}+\cdots=\infty$), it suffices to show that the sum of inverses of all natural numbers missing at least one digit in base b is convergent and we will reach a contradiction. But of course, it suffices to prove it for a fixed (but arbitrary) digit j. For any $n \geq 1$, there are at most $(b-1)^n$ numbers which have n digits in base b, all different from j. Thus, since each one of them is at least equal to b^{n-1}, the sum of inverses of numbers that miss the digit j when written in base b is at most equal to $\sum_n b\left(\frac{b-1}{b}\right)^n$, which converges. The conclusion follows.

We return to classical problems to discuss a beautiful problem, that appeared in the Tournament of the Towns in 1982, in a Russian Team Selection Test in 1997 and also in the Bulgarian Olympiad in 2003. It's beauty explains probably the preference for this problem.

Example 8. Let $f \in Z[X]$ be a polynomial with leading coefficient 1 such that for any natural number n the equation $f(x) = 2^n$ has at least one natural solution. Prove that $\deg f = 1$.

Solution. So, the problem states that there exists a sequence of positive integers $(x_n)_{n\geq 1}$ such that $f(x_n) = 2^n$. Let us suppose that $\deg f = k > 1$. Then, for large values of x, $f(x)$ behaves like x^k. So, trying to find the right convergent sequence, we could try first to "think big": we have $x_n^k \cong 2^n$, that is for large n, x_n behaves like $2^{\frac{n}{k}}$. Then, a good possibly convergent sequence could be $x_{n+k} - 2x_n$. Now, the hard part: proving that this sequence is indeed convergent. First, we will show that $\frac{x_{n+k}}{x_n}$ converges to 2. This is easy, since the relation

$f(x_{n+k}) = 2^k f(x_n)$ implies

$$\frac{f(x_{n+k})}{x_{n+k}^k}\left(\frac{x_{n+k}}{x_n}\right)^k = 2^k \cdot \frac{f(x_n)}{x_n^k}$$

and since

$$\lim_{x\to\infty}\frac{f(x)}{x^k} = 1 \text{ and } \lim_{n\to\infty} x_n = \infty$$

(do you see why?), we find that indeed

$$\lim_{n\to\infty}\frac{x_{n+k}}{x_n} = 2.$$

We will see that this will help us a lot. Indeed, let us write

$$f(x) = x^k + \sum_{i=0}^{k-1} a_i x^i.$$

Then $f(x_{n+k}) = 2^k f(x_n)$ can be also written

$$x_{n+k} - 2x_n = \frac{\displaystyle\sum_{i=0}^{k-1} a_i(2^k x_n^i - x_{n+k}^i)}{\displaystyle\sum_{i=0}^{k-1}(2x_n)^i x_{n+k}^{k-i-1}}$$

But from the fact that $\lim\limits_{n\to\infty}\frac{x_{n+k}}{x_n} = 2.$ it follows that the right-hand side in the above relation is also convergent. So, $(x_{n+k} - 2x_n)_{n\geq 1}$ is convergent and it follows that there exist M, N such that for all $n \geq M$ we have $x_{n+k} = 2x_n + N$. But now the problem is almost done, since the last result combined with $f(x_{n+k}) = 2^k f(x_n)$ yields $f(2x_n + N) = 2^k f(x_n)$ for $n \geq M$, that is $f(2x + N) = 2^k f(x)$. So, an arithmetical property of the polynomial turned into an algebraic one using analysis. This algebraic property helps us to immediately solve the problem. Indeed, we see that if z is a complex root of f, then $2z + N, 4z + 3N, 8z + 7N, \dots$ are all roots of f. Since f is non-zero, this sequence must be finite and this can happen only for $z = -N$. Since $-N$ is the only root of f, we deduce that $f(x) = (x + N)^k$. But since the equation $f(x) = 2^{2k+1}$ has natural roots, we find that $2^{\frac{1}{k}} \in N$, which implies, contradiction. Thus, our assumption was wrong and $\deg f = 1$.

The idea of the following problem is so beautiful, that after reading the solution the reader will have the impression that the problem is trivial. Wrong! The problem is really difficult and to make again an experiment, we will ask the reader to struggle a lot before reading the solution. He will see the difficulty.

Example 9. Let $\pi(n)$ be the number of prime numbers smaller than or equal to n. Prove that there exist infinitely many numbers n such that $\pi(n)|n$.

AMM

Solution. First, let us prove the following result, which is the key of the problem.

Lemma. *For any increasing sequence of positive integers* $(a_n)_{n\geq 1}$ *such that* $\lim_{n\to\infty} \frac{a_n}{n} = 0$*, the sequence* $\left(\frac{n}{a_n}\right)_{n\geq 1}$ *contains all natural numbers. In particular, for infinitely many* n *we have that* n *divides* a_n.

Proof. Even if it seems unbelievable, this is true and moreover the proof is extremely short. Let $m \geq 1$ be a natural number. Consider the set $A = \left\{n \geq 1 | \ \frac{a_{mn}}{mn} \geq \frac{1}{m}\right\}$. This set contains and it is bounded, since $\lim_{n\to\infty} \frac{a_{mn}}{mn} = 0$. Thus it has a maximal element k. If $\frac{a_{mk}}{mk} = \frac{1}{m}$, then m is in the sequence $\left(\frac{n}{a_n}\right)_{n\geq 1}$. Otherwise, we have $a_{m(k+1)} \geq a_{mk} \geq k+1$, which shows that $k+1$ is also in the set, contradiction with the maximality of k. The lemma is proved.

Thus, all we need to show is that $\lim_{n\to\infty} \frac{\pi(n)}{n} = 0$. Fortunately, this is well-known and not difficult to prove. There are easier proofs than the following one, but we prefer to deduce it from a famous and beautiful result of Erdos.

Erdos's theorem. *We have* $\prod_{\substack{p\le n\\ p\, prime}} p \le 4^n$.

The proof of this result is magnificient. The proof is by induction. For small values of n it is clear. Now, assume the inequality true for all values smaller than n and let us prove that $\prod_{\substack{p\le n\\ p\, prime}} p \le 4^n$. If $n$is even, we have nothing to prove, since

$$\prod_{\substack{p\le n\\ p\, prime}} = \prod_{\substack{p\le n-1\\ p\, prime}} p \le 4^{n-1} < 4^n.$$

Now, assume that $n = 2k+1$ and consider the binomial coefficient

$$\binom{2k+1}{k} = \frac{(k+2)\dots(2k+1)}{k!}.$$

A simple application of the fact that

$$2^{2k+1} = \sum_{i\ge 0}\binom{2k+1}{i}$$

shows that

$$\binom{2k+1}{k} \le 4^k.$$

Thus, using the inductive hypothesis, we find that

$$\prod_{\substack{p\le n\\ p\, prime}} p \le \prod_{\substack{p\le k+1\\ p\, prime}} p \prod_{\substack{k+2\le p\le 2k+1\\ p\, prime}} p \le 4^{k+1}\cdot 4^k = 4^n.$$

Now, the fact that $\lim_{n\to\infty}\frac{\pi(n)}{n} = 0$ is trivial. Indeed, fix $k \ge 1$. We have for all large n the inequality

$$n\lg 4 \ge \sum_{\substack{k\le p\le n\\ p\, prime}} \lg p \ge \lg k(\pi(n) - \pi(k)),$$

which shows that

$$\pi(n) \le \frac{\pi(k)}{n} + \frac{\lg 4}{\lg k}.$$

This shows of course that $\lim\limits_{n\to\infty} \dfrac{\pi(n)}{n} = 0$. The problem is solved.

It is time now for the last problem, which is, as usual, very hard. We don't exaggerate if we say that the following problem is exceptionally difficult.

Example 10. Let $a, b > 1$ be natural numbers such that for any natural number n, $a^n - 1 | b^n - 1$. Prove that b is a natural power of a.

Marius Cavachi, AMM

Solution. This time we will be able to find the right convergent sequence only after some double recurrences. Let us see. So, initially we are given that there exists a sequence of positive integers $(x_n^1)_{n\geq 1}$ such that $x_n^1 = \dfrac{b^n - 1}{a^n - 1}$ Then, $x_n^1 \cong \left(\dfrac{b}{a}\right)^n$ for large values of n. So, we could expect that the sequence $(x_n^2)_{n\geq 1}$, $x_n^2 = bx_n^1 - ax_{n+1}^2$ is convergent. Unfortunately,

$$x_n^2 = \frac{b^{n+1}(a-1) - a^{n+1}(b-1) + a - b}{(a^n - 1)(a^{n+1} - 1)},$$

which is not necessarily convergent. But... if we look again at this sequence, we see that for large values of n it grows like $\left(\dfrac{b}{a^2}\right)^n$, so much slower. And this is the good idea: repeat this procedure until the final sequence behaves like $\left(\dfrac{b}{a^{k+1}}\right)^n$, where k is chosen such that $a^k \leq b < a^{k+1}$. Thus, the final sequence will converge to 0. Again, the hard part has just begun, since we have to prove that if we define $x_n^{i+1} = bx_n^i - a^i x_{n+1}^i$ then $\lim\limits_{n\to\infty} x_n^{k+1} = 0$. This isn't easy at all. The idea is to compute x_n^3 and after that to prove the following statement: for any $i \geq 1$ the sequence $(x_n^i)_{n\geq 1}$ has the form

$$\frac{c_i b^n + c_{i-1} a^{(i-1)n} + \cdots + c_1 a^n + c_0}{(a^{n+i-1} - 1)(a^{n+i-2} - 1)\ldots(a^n - 1)}$$

for some constants $c_0, c_1, \ldots, c_i$. Proving this is not so hard, the hard part was to think about it. How can we prove the statement otherwise than by induction? And induction turns out to be quite easy. Supposing that the statement is true for i, then the corresponding statement for $i+1$ follows from $x_n^{i+1} = bx_n^i - a^i x_{n+1}^i$ directly (note that to make the difference, we just have to multiply the numerator $c_i b^n + c_{i-1} a^{(i-1)n} + \cdots + c_1 a^n + c_0$ with b and $a^{n+i} - 1$. Then, we proceed in the same way with the second fraction and the term $b^{n+1} a^{n+i}$ will vanish). So, we have found a formula which shows that as soon as $a^i > b$ we have $\lim\limits_{n\to\infty} x_n^i = 0$. So, we have deduced that $\lim\limits_{n\to\infty} x_n^{k+1} = 0$. Another step of the solution is to take the minimal index j such that $\lim\limits_{n\to\infty} x_n^j = 0$. Obviously, $j > 1$ and the recurrence relation $x_n^{i+1} = bx_n^i - a^i x_{n+1}^i$ shows that $x_n^i \in Z$ for all n, i. Thus, there exists M such that whenever $n \geq M$ we have $x_n^j = 0$. This is the same as $bx_n^{j-1} = a^j x_{n+1}^{j-1}$ for all $n \geq M$, which implies $x_n^{j-1} = \left(\dfrac{b}{a^j}\right)^{n-M} x_M^{j-1}$ for all $n \geq M$. Let us suppose that b is not a multiple of a. Since $\left(\dfrac{b}{a^j}\right)^{n-M} x_M^{j-1} \in Z$ for all $n \geq M$, we must have $x_M^{j-1} = 0$ and so $x_n^{j-1} = 0$ for $n \geq M$, which means $\lim\limits_{n\to\infty} x_n^j = 0$. But this contradicts the minimality of j. Since we have reached a contradiction, we must have $a|b$. Let us write $b = ca$. Then, the relation $a^n - 1|b^n - 1$ implies $a^n - 1|c^n - 1$. And now are finally done. Why? We have just seen that $a^n - 1|c^n - 1$ for all $n \geq 1$. But our previous argument applied for c instead of b shows that $a|c$. Thus, $c = ad$ and we deduce again that $a|d$. Since this process cannot be infinite, b must be a power of a.

It worth saying that there exist an even stronger result: it is enough to suppose that $a^n - 1|b^n - 1$ for an infinite number n, but this is a much more difficult problem. It follows from a result found by Bugeaud, Corvaja and Zannier in 2003:

If $a, b > 1$ are multiplicatively independent in Q^* (that is $\log_a b \notin Q$), then for any $\varepsilon > 0$ there exists $n_0 = n_0(a, b, \varepsilon)$ such that $gcd(a^n - 1, b^n -$

$1) < 2^{\varepsilon n}$ for all $n \geq n_0$. Unfortunately, the proof is too advanced to be presented here.

Problems for training

1. Let $f \in Z[X]$ be a polynomial of degree k such that for all $n \in \mathbb{N}$ we have $\sqrt[k]{f(n)} \in Z$. Prove that there exists integer numbers a, b such that $f(x) = (ax+b)^k$.

2. Find all arithmetic progressions of positive integers $(a_n)_{n\geq 1}$ such that for all $n \geq 1$ the number $a_1 + a_2 + \cdots + a_n$ is a perfect square.

Laurentiu Panaitopol, Romanian Olympiad 1991

3. Let p be a polynomial with integer coefficients such that there exists a sequence of pair wise distinct positive integers $(a_n)_{n\geq 1}$ such that $p(a_1) = 0$, $p(a_2) = a_1$, $p(a_3) = a_2, \ldots$. Find the degree of this polynomial.

Tournament of the Towns, 2003

4. Let $f, g : \mathbb{N}^* \to \mathbb{N}^*$ two functions such that $|f(n) - n| \leq 2004\sqrt{n}$ and $n^2 + g^2(n) = 2f^2(n)$. Prove that if f or g is surjective, then these functions have infinitely many fixed points.

Gabriel Dospinescu, Moldova TST 2004

5. Let a, b be natural numbers such that for any natural number n, the decimal representation of $a + bn$ contains a sequence of consecutive digits which form the decimal representation of n (for example, if $a = 600$, $b = 35$, $n = 16$ we have $600 + 16 \cdot 35 = 1160$). Prove that b is a power of 10.

Tournament of the Towns, 2002

6. Let $a, b > 1$ be positive integers. Prove that for any given $k > 0$ there are infinitely many numbers n such that $\varphi(an + b) < kn$, where φ is the Euler totient function.

Gabriel Dospinescu

7. Let b an integer at least equal to 5 and define the number $x_n = \underbrace{11\ldots1}_{n-1}\underbrace{22\ldots2}_{n}5$ in base b. Prove that x_n is a perfect square for all sufficiently large n if and only if $b = 10$.

Laurentiu Panaitopol, IMO Shortlist 2004

8. Find all triplets of integer numbers a, b, c such that for any positive integer n, $a \cdot 2^n + b$ is a divisor of $c^n + 1$.

Gabriel Dospinescu

11. Suppose that a is a real number such that all numbers $1^a, 2^a, 3^a, \ldots$ are integers. Then prove that a is also integer.

Putnam

12. Find all complex polynomials f having the property: there exists $a \geq 2$ a natural number such that for all sufficiently large n, the equation $f(x) = a^{n^2}$ has at least a positive rational solution.

Gabriel Dospinescu, Revue de Mathematiques Speciales

13. Let f be a complex polynomial having the property that for all natural number n, the equation $f(x) = n$ has at least a rational solution. Then f has degree at most 1.

Mathlinks Contest

14. Let A be a set of natural numbers, which contains at least one number among any 2006 consecutive natural numbers and let f a non-constant polynomial with integer coefficients. Prove that there exists a number N such that for any $n \geq N$ there are at least $\sqrt{\ln \ln n}$ different prime numbers dividing the number $\prod_{\substack{N \leq k \leq n \\ k \in A}} f(k)$.

Gabriel Dospinescu

15. Prove that in any strictly increasing sequence of positive integers $(a_n)_{n\geq 1}$ which satisfies $a_n < 100n$ for all n, one can find infinitely many terms containing at least 1986 consecutive 1.

Kvant

16. Any infinite arithmetical progression contains infinitely many terms that are not powers of integers.

17. Find all a, b, c such that for all sufficiently large n, the number $a \cdot 4^n + b \cdot 6^n + c \cdot 9^n$ is a perfect square.

18. Let f, g two real polynomials of degree 2 such that for any real x, if $f(x)$ is integer, so is $g(x)$. Then there are integers m, n such that $g(x) = mf(x) + n$ for all x.

Bulgarian Olympiad

19. Try to generalize the preceding problem (this is for the die-hards!!!).

20. Find all pairs of natural numbers a, b such that for every positive integer n the number $an + b$ is triangular if and only if n is triangular.

After a Putnam problem

21. Let $(a_n)_{n\geq 1}$ be an infinite and strictly increasing sequence of positive integers such that for all $n \geq 2002$, $a_n | a_1 + a_2 + \cdots + a_{n-1}$. Prove that there exists n_0 such that for all $n \geq n_0$ we have $a_n = a_1 + a_2 + \cdots + a_{n-1}$.

Tournament of the Towns, 2002

22. Find all real polynomials such that the image of any repunit is also a repunit.

After a problem from Kvant

23. Fie doua multimi finite de numere reale pozitive cu proprietatea ca

$$\left\{\sum_{x\in A} x^n \mid n \in \mathbb{R}\right\} \subset \left\{\sum_{x\in B} x^n \mid n \in \mathbb{R}\right\}.$$

Sa se arate ca exista $k \in \mathbb{R}$ astfel incat $A = \{x^k \mid x \in B\}$.

Gabriel Dospinescu