Lemmas In Olympiad Geometry

Navneel Singhal

July 14, 2016

Geometry is the art of correct reasoning from incorrectly drawn figures.

- Henri Poincaré

1 Introduction

Here is a collection of some useful lemmas in geometry, some of them well known, some obscure and some by the author himself. This list of lemmas is also intended to be a list of some easier problems and also as some configurations that frequently appear on contests. Usually these lemmas will be intermediate results that will help you reach the solution in many cases, and maybe even trivialize the problem. These will help you write some really elegant solutions (and will also help you to simplify your bashes in cases of some problems that don't yield easily to synthetic solutions.) So have fun proving these lemmas and using them to the fullest advantage in your Olympiad journey!

2 Some Notations

- By (XYZ) we denote the circumcircle of ΔXYZ, by (XY) the circle with XY as diameter, and by (M, r) the circle with centre M and radius r, the radius being dropped when the context is clear.
- We denote H as the orthocentre, O as the circumcentre, G as the centroid, N as the nine-point centre, I as the incentre, N_a as the Nagel point and G_e as the Gergonne point of ΔABC

3 Orthocenter related properties

- O and H are isogonal conjugates.
- Reflections of the orthocenter over the sides and the midpoints of the sides lie on (ABC).

- The O is the orthocenter of the medial triangle. a consequence is that AH=$2OM$.
- If DEF is the orthic triangle of $\triangle ABC$ then $\{H, A, B, C\}$ is the set of in/excenters of $\triangle DEF$.
- Let A', B' and C' be the reflections of H over the sides BC, CA and AB respectively. Then A is the midpoint of arc $B'C'$ and so on.
- Let X and Y be the orthocenters of the triangles ABC and DBC of cyclic quadrilateral $ABCD$. Then $BCXY$ is a parallelogram.
- Simson line of a point P on (ABC) bisects PH.(can be quoted)
- Let $ABCD$ be a complete quadrilateral. $P = AD \cap BC$ and $Q = AB \cap CD$. Then circles with diameter AC, BD and PQ are coaxial and their radical axis is the line joining the orthocenters of the triangles PAB, PCD, QAD and QBC.(Gauss - Bodenmiller's theorem). This line is perpendicular to the line joining the midpoints of PQ, AC and BD which is called the Steiner line of the complete quadrilateral $ABCD$.
- Let AA_1, BB_1 and CC_1 be three cevians. Then H is the radical center of the circles with diameters as AA_1, BB_1 and CC_1.

4 Symmedian Related Properties

Let the tangents to (ABC) at B and C meet at X. Let AX meet (ABC) at K and BC at J. Then the following properties hold:

- $\frac{BJ}{JC} = (\frac{AB}{AC})^2$
- Quadrilateral $BCAK$ is harmonic.
- $\triangle ABK \sim \triangle AMC$
- (AO) and (BOC) meet on the midpoint of AK.
- BC is a symmedian in each of $\triangle BAK$ and $\triangle CAK$.
- BC is also an angle bisector of $\angle AMK$ and MK is its external angle bisector.
- Tangents at A and K to (ABC) meet on BC.

Let the reflection of K in BC be Y. Then

- Y is the image of M (midpoint of BC) under the inversion centered at A which sends H to the foot of the A-altitude.

- $(BHC), (AH), AM$, circles tangent to BC at B and C and passing through A, and the circumcircle of the triangle formed by the midpoint of AH, M and the circumcenter of BHC (the reflection of O in BC), all pass through Y. The last triangle mentioned forms an isosceles trapezoid alongwith Y.
- The symmedians concur at the Symmedian point which is the unique point such that it is the centroid of its pedal triangle with respect to ABC.
- Let the foot of the A-altitude on BC be D and on (ABC) be A', and midpoint of BC be M. Then $(DA'M)$ cuts (ABC) again at K, the intersection of the A-symmedian with the circumcircle.
- Let the foot of the $A-$ and the $B-$ altitudes on the sides be D and E. Then AK, DE, B-midline and the line antiparallel to BC through B with respect to $\triangle ABC$ are concurrent.
- Let K' be the symmedian point of $\triangle ABC$ and Q be the midpoint of the A-altitude. Then $K'Q$ bisects BC.

5 In/excenter related properties

Let the touchpoint of the incircle with BC, CA, AB be D, E and F respectively. The A-excircle touches the side BC at D'.

- (Diameter of incircle) Let the diameter of the incircle through D be DD''. Then A, D'' and D' are collinear and D and D' are isotomic points with respect to side BC.
- (Diameter of excircle)Let the diameter of the excircle through D' be $D'D_1$. Then A, D and D_1 are collinear.
- (Fact 5) The midpoints of the arcs BC, CA and AB are equidistant from the incenter, respective endpoints of the arcs and the respective excenter.
- The diameter of the incircle DD'', the A-median and the A-touch-chord of the incircle are concurrent.
- Let the A-touch chord of the incircle be EF with E on AC and F on AB. Then the B-midline, C-bisector, EF and the circle with BC as diameter concur at X such that $FXIDB$ is a cyclic pentagon.
- Let I_A, I_B and I_C be the excenters, M_A, M_B and M_C be the arc midpoints of the arcs of (ABC) cut off by the sides containing exactly 2 vertices, M'_A, M'_B and M'_C be the antipodes of M_A, M_B and M_C respectively. Then I is the orthocenter of $I_AI_BI_C$. M'_A, M'_B and M'_C are the midpoints of the sides of $\triangle I_AI_BI_C$ and $\triangle ABC$ is the orthic triangle of $\triangle I_AI_BI_C$, with (ABC) being the nine-point circle of $\triangle I_AI_BI_C$. Also, I_CI_BCB is cyclic with I_CI_B as the diameter. The contact triangle is also homothetic with $\triangle I_AI_BI_C$.

- DI_A, $D'I$ concur on the midpoint of the A-altitude.
- Let EF meet BC at Z. Then $IK \perp AD$ and K is the pole of the line AD with respect to the incircle. Also, $\{B, C, D, K\}$ is harmonic. Consequences include: The tangent to the incircle at $AD \cap (I)$ passes through X, AD bisects $\angle B(AD \cap IK)C$.
- Antipode of A in (ABC), the foot of the D-altitude in the contact triangle and I lie on a line which passes through the second intersection of (ABC) and $(AEIF)$.
- OI is the Euler line of the contact triangle.
- $\angle AM'_A I = \angle IMB$ where M is the midpoint of BC.
- (Curvilinear Incircles) A curvilinear incircle ω is a circle tangent to an arc of (ABC) containing all three vertices of ABC and also tangent to one of its sides. For the sake of definiteness we assume that the circle is tangent to the side AB at K, to arc ACB at T, and CD is a cevian tangent to ω at L. Then the following hold true:
  1. M_C, K, T are collinear and $M_C K.M_C T = M_C B^2 = M_C A^2$
  2. $CILT$ is a cyclic quadrilateral.
  3. $I \in KL$
- (Mixtilinear Incircles) The A-mixtilinear incircle ω_A is the circle tangent to AB and AC and (ABC) at K, L, T respectively. Then the following hold true:
  1. I is the midpoint of KL.
  2. TI passes through M'_A, and is isogonal to TA in the triangle TKL (look for symmedians).
  3. $BKIT$ and $CLIT$ are cyclic, with CI and BI being tangent to them..
  4. AT and AD' are isogonal in $\triangle ABC$. AT and DT are isogonal in $\triangle TBC$.
  5. AT passes through the insimilicenter of the incircle and (ABC) which is on OI.
  6. The circumcircle of the triangle formed by the intersections of the A-bisector and BC and (ABC) and D passes through T.
  7. BC, KL and TM_A are concurrent.
  8. $\angle ATM_C = \angle M_B TI$
  9. TM_A and AD meet on the mixtilinear incircle.
  10. AD' and TI meet on the mixtilinear incircle.
  11. ω_A is the image of the A-excircle under the composition of an inversion at A with radius $\sqrt{bc}$ and a reflection over the angle bisector of $\angle BAC$.

- If Y and Z are the feet of the B- and the C- bisectors on the opposite sides then $OI_A \perp YZ$.
- A circle through A cuts the sides AB and AC at B' and C' such that $BB' = CC'$ iff it passes through M'_A.
- The medial triangle and the triangle homothetic to $\triangle ABC$ at N_a with ratio $\frac{1}{2}$ share a common incircle.
- N_a is the incenter of the antimedial triangle, or we can say that I is the Nagel point of the medial triangle.

6 Feet of the altitudes and the midpoints

Here we denote by M the midpoint of BC and by D the foot of the A-altitude.

- Reflection of A in N coincides with the reflection of O in M.
- (AH), (AM), (ABC) and HM concur at a point. The midpoints of AH and AM are collinear with O.
- Ray $DG \cap (ABC)$ forms an isosceles trapezoid with ABC.
- Let P and Q be the midpoints of AB and AC and the feet of B- and C-altitudes be E and F. Let H' be the midpoint of AH. Then $PDMQ$ is an isosceles trapezoid, H' and M are the midpoints of the arcs formed by E and F in the nine-point circle.
- The triangle formed by the reflections of A, B and C over D, E and F is homothetic with the triangle formed by the reflections of N over the sides at O with ratio 2. (In fact, this is equivalent to the fact that the reflections of H over the sides lie on the circumcircle and also with the fact that $AH{=}2OM$. Prove this!).
- Let E be the foot of the B-altitude and P be the point on AC such that BP is antiparallel to BC wrt $\triangle ABC$. Then the A-symmedian, B-midline, DE and BP concur at a point.

7 Triangle centers

- (Nagel Line) G, I and N_a are collinear, and $GN_a = 2IG$.
- (Isogonal Conjugate) Q is said to be the **isogonal conjugate** of a point P not on the sides of $\triangle ABC$ if PA and QA are reflections of each other in the angle bisector of $\angle A$, PB and QB are reflections of each other in the angle bisector of $\angle B$, and PC and QC are reflections of each other in the angle bisector of $\angle C$.

- The triangle formed by the reflections of a point P in the sides of $\triangle ABC$ has the circumcenter as the isogonal conjugate of P. Also the circumcircles of the triangle of reflections as (described above) of a point and its isogonal conjugate are congruent. Consequences: the pedal triangles (triangle formed by feet of perpendiculars from a point to the sides of $\triangle ABC$) of a point and its isogonal conjugate are concyclic, the center is the midpoint of the line segment joining them, Simson's line, and also the existence of the isogonal conjugate.

- If P and Q are isogonal conjugates, then PA, PB and PC are perpendicular to the sides of the pedal triangle of Q.

- The cevian triangle of a point P with respect to ABC is the triangle formed by the points of intersection of the lines joining P to vertices and the opposite sides. The **isotomic conjugate** of a point P is defined as the point of intersection of the cevians formed by a vertex and the reflection of the corresponding vertex of the cevian triangle in the midpoint of the corresponding side of $\triangle ABC$. Then the cevian triangles of a point and its isotomic conjugate have the same area.

- N_a is the isotomic conjugate of G_e.

- The isogonal conjugate of N_a is the insimilicenter of the incircle and the circumcircle, and that of G_e is the exsimilicenter of the incircle and the circumcircle.

- (Brocard Points) The first Brocard point is the point Q_1 such that $\angle Q_1AB = \angle Q_1BC = \angle Q_1CA$. The second Brocard point is the point Q_2 such that $\angle Q_2BA = \angle Q_2CB = \angle Q_2AC$. Note that they are isogonal conjugates. Let the common angle be ω. Then a useful result is that $\cot\omega = \cot A + \cot B + \cot C$.

- (Feuerbach point) The nine-point circle of a non-equilateral triangle is tangent to the incircle and the excircles. The point of tangency with the incircle is called the Feuerbach point of the triangle.

- (Poncelet point) For a quadruple $\{W, X, Y, Z\}$ of points, the nine point circles of the triangles formed by points in this set, the pedal circles of the points with respect to the triangle formed by the other three are concurrent at the **Poncelet point** of the quadruple. Note that the Feuerbach point is the Poncelet point of the quadruple $\{A, B, C, I\}$.

8 Miscellaneous useful properties

Note that some of the properties mentioned out here are theorems and not lemmas.

- The locus of the geometric transform is the transform of the locus. (incredibly useful)

- (Spiral Similarity Lemma) Let AB and CD be two segments and let $AC \cap BD = X$. Let (ABX) and (CDX) intersect again at Y. Then Y is the center of spiral similarity sending AB to CD and sending AC to BD. (useful)

- (Three Homotheties) The center of the composition of 2 homotheties lies on the line joining centers of them. (useful)

- The center of inversion swapping 2 circles is collinear with their centers. (useful)

- Let $pow(O, \omega)$ be the power of a point O with respect to the circle ω. Then the locus of points such that $\frac{pow(P,\omega_1)}{pow(P,\omega_2)}$ is constant is a circle coaxial with ω_1 and ω_2.

- See Monge's theorem, Monge - deAlembart's theorem. (Hint for proof: Think 3D, think cones).

- (Cevian Nest) If $A_1B_1C_1$ is the cevian triangle of a point P wrt ABC and if $A_2B_2C_2$ is the cevian triangle of Q wrt $A_1B_1C_1$ then AA_2, BB_2 and CC_2 concur at a point.

- (Apollonius circles) The circle with ends of the diameter as the feet of the A-bisectors onto BC is the locus of points P such that $\frac{PB}{PC} = \frac{AB}{AC}$.

- See Brocard's Theorem (about the self polarity of the triangle formed by intersections of opposite sides of a cyclic quadrilateral).

- (Miquel Point) Let P, Q, R be points on BC, CA, AB respectively. Then $(AQR), (PBR), (PQC)$ concur at a point M. The centers of the circles form a triangle similar to $\triangle ABC$.

- (Also Miquel Point) Let $ABCD$ be a quadrilateral. $AB \cap CD = P, AD \cap BC = Q, AC \cap BD = R$. Then $(PAD), (PBC), (QAB), (QCD)$ concur at M which is called the Miquel point M. It is concyclic with the centers of the circles. M is the center of spiral similarity sending AB to CD and BC to DA. If $ABCD$ is cyclic, then many beautiful properties emerge. Let the center be O.

    1. (OAC) and (OBD) also pass through M.
    2. M is the inverse of R in (O) and thus lies on PQ, and is the foot of (O) on PQ.
    3. MO bisects $\angle AMC$ and $\angle BMD$.

- See butterfly theorem and its generalizations.

- Given any quadrilateral $ABCD$ and the midpoints X, Y, Z, W, U, V of AB, BC, CD, DA, AC, BD, and the centroids G_1, G_2, G_3, G_4 of the triangles BCD, CDA, DAB, ABC then $XZ, YW, UV, AG_1, BG_2, CG_3, DG_4$

are all concurrent at a point P which bisects the first three segments and divides the last four in a ratio 3:1.

- If in the above lemma the quadrilateral is cyclic and the orthocentres of the triangles BCD, CDA, DAB, ABC are H_1, H_2, H_3, H_4 respectively then AH_1, BH_2, CH_3, DH_4 are concurrent at the reflection of the centre of the circle in P, say Q. Also Q is the midpoint of each of these segments.

- If a line makes equal angles with the opposite sides of a cyclic quadrilateral, then circles can be drawn tangent to each pair, where this line meets them, and these circles are coaxial with the original circle.

- Given an angle and a circle through the vertex of the angle, cutting its bisector at a fixed point. Then the sum of the intercepts of the circle on the sides of the angle is invariant.

- (Utkarsh's Isogonal Line Lemma) AX and AY are isogonal wrt $\angle A$. BX meets CY in Z_1 and BY meets CX in Z_2. then AZ_1, AZ_2 are isogonal wrt $\angle A$.

- Let ω_1 and ω_2 be two circles with the center of ω_1 lying on ω_2. Then the inverse of ω_1 with respect to ω_2 is their radical axis.

9 General tips and tricks

Here are some general tips and tricks which were found to be useful to the author during Olympiads.

- Create phantom points: This means creating a point which satisfies a property that is to be proved and showing that it is unique and satisfies the problem condition.

- Negative inversion at H sending the nine-point circle to the circumcircle.

- Inversion about the circumcircle.

- "Maximum irritating circles" inversion: try to get rid of as many irritating circles you can (this is of course based on your intuition).

- Inversion about the incircle (quite useful for many problems, most of them involving the contact triangle and (ABC)).

- Inversion at A. This one sends the circumcenter to the reflection of A in the line joining the inverses of B and C. Inversion at A is most useful (as far as I think) in the following 2 cases:

  1. Inversion with radius $\sqrt{AH.AD}$ where D is the $AH \cap BC$. This one sends the foot of the B-altitude to C, foot of the C-altitude to B and H to D.

2. The composition of an inversion with radius $\sqrt{bc}$ and flip over AI. This is quite famous and preserves the original triangle and is incredibly powerful for overlays. It sends BC to the circumcircle of ABC.

- To show two circles are orthogonal, you may show that the inverse of one circle in the other is itself, or try to show that the endpoints of a diameter of a circle lie on the polars of one another with respect to the other circle.

- To show that 2 circles are tangent, the following can be used:

  1. Show that they are homothetic at a point lying on one of the circles, usually done by finding 2 triangles homothetic at a point on one of the circles, having their circumcircles as the given circles.
  2. Or draw a tangent to the circle at an intersection point (after showing that they intersect!) and then show that it is tangent to the other circle too.
  3. Or show that the distance between the centers is either the sum or the difference of their radii.

- Another useful trick worth mentioning is point circles. They are especially useful for radical axes, because the radical axis of 2 point circles is their perpendicular bisector, and the radical axis of a point circle P and a non-point circle is the P-midline of the triangle formed by P and the contact points of the tangents from P to the circle.

10 References

1. Yufei Zhao's Notes http://yufeizhao.com/olympiad.html

2. Darij Grinberg's website http://web.mit.edu/ darij/www/

3. Evan Chen's book "Euclidean Geometry in Mathematical Olympiads"

4. I.F. Sharygin "Problems in Plane geometry"

5. Mathematical Reflections Journal

6. "107 Geometry Problems" - Titu Andreescu

7. Various posts and blogs at http://artofproblemsolving.com

8. Discussions at IMOTC-2016 India