Multiple Choice Questions

Question 1. Write 2013 as a sum of \( m \) prime numbers. The smallest value of \( m \) is:

(A): 2; (B): 3; (C): 4; (D): 1; (E): None of the above.

Answer: (A).

Since \( 2013 = 3 \times 671 \) then 2013 is not a prime number. Hence \( m \geq 2 \). On the other hand, \( 2013 = 2 + 2011 \) as a sum of 2 prime numbers. Thus, the smallest value of \( m \) is 2.

Question 2. How many natural numbers \( n \) are there so that \( n^2 + 2014 \) is a perfect square.

(A):1; (B):2; (C):3; (D):4; (E) None of the above.

Answer: (E).

Suppose that \( n^2 + 2014 \) is a perfect square, i.e. \( n^2 + 2014 = m^2 \), where \( m \in \mathbb{N}^* \). It follows \( (m-n)(m+n) = 2014 \) and then at least one of \( m-n \) and \( m+n \) is even. Since \( (m-n)+(m+n) = 2m \) is even then both \( m+n \) and \( m-n \) are even. Hence \( (m-n) \times (m+n) \) is divisible by 4. It is impossible for 2014 is not divisible by 4. Thus, there are no natural numbers \( n \) so that \( n^2 + 2014 \) is a perfect square.

Question 3. The largest integer not exceeding \( [(n+1)\alpha] - [n\alpha] \), where \( n \) is a natural number, \( \alpha = \frac{\sqrt{2013}}{\sqrt{2014}} \), is:

(A):1; (B):2; (C):3; (D):4; (E) None of the above.

Answer: (E).

Let \( a_n = [(n+1)\alpha] - [n\alpha] \), for \( n = 0, 1, 2, \ldots \) From the inequalities \( 0 \leq a_n \leq [n\alpha+1] - [n\alpha] = 1 \) for every natural number \( n \) and \( a_n \) is an integer, it follows \( a_n = [(n+1)\alpha] - [n\alpha] \in \{0, 1\} \) for every \( n \in \mathbb{N} \). We prove that 0 is the largest integer not exceeding every \( [(n+1)\alpha] - [n\alpha] \). Indeed,
for \( n = 0 \), we find \( a_0 = [\alpha] = 0 \). Hence, the largest integer not exceeding \( [(n + 1)\alpha] - [n\alpha] \), where \( n \) is a natural number and \( \alpha = \frac{\sqrt{2013}}{\sqrt{2014}} \) must be 0.

**Question 4.** Let \( A \) be an even number but not divisible by 10. The last two digits of \( A^{20} \) are:

(A): 46; (B): 56; (C): 66; (D): 76; (E): None of the above.

**Answer: (D).**

Since \( A \) is even then \( A = 2n, n \in \mathbb{N} \). It follows \( A^{20} = (2n)^{20} = (4n^2)^{10} \Rightarrow A^{20} \vdots 4 \).

On the other hand, \( A \) is not divisible by 10, \[
\begin{align*}
A &= 5k \pm 1 \\
A &= 5k \pm 2
\end{align*}
\]

If \( A = 5k \pm 1 \) then \( A^{20} = (5k \pm 1)^{20} = (5k)^{20} + 20(5k)^{19} + 20(5k)^{18} + \ldots + 20.5k + 1 \), hence \( A^{20} \equiv 1 \) (mod 25).

If \( A = 5k \pm 2 \), then \( A^{20} = 25q + 2^{20} = 25q + (1025 - 1)^2 \), hence \( A^{20} \equiv 1 \) (mod 25).

Thus, \( A^{20} \equiv 1 \) (mod 25) for every \( A \) and the last two digits of \( A^{20} \) are in \{01; 26; 51; 76\}. Since \( A^{20} \) is divisible by 4 then the last two digits of \( A^{20} \) are 76.

**Question 5.** The number of integer solutions \( x \) of the equation below

\[(12x - 1)(6x - 1)(4x - 1)(3x - 1) = 330.\]

is: (A): 0; (B): 1; (C): 2; (D): 3; (E): None of the above.

**Answer: (B).**

Multiply both sides of the equation by 2.3.4, we find

\[(12x - 1)(12x - 2)(12x - 3)(12x - 4) = 11 \times 10 \times 9 \times 8.\]

Left side is the product of 4 non-zero consecutive integers then all factors are the same sign. This argument follows that

\[
\begin{align*}
(12x - 1)(12x - 2)(12x - 3)(12x - 4) &= 11 \times 10 \times 9 \times 8 \\
(12x - 1)(12x - 2)(12x - 3)(12x - 4) &= (-11) \times (-10) \times (-9) \times (-8)
\end{align*}
\]

The 1st equation has a root \( x = 1 \), the 2nd equation has no integer roots.

**Question 6.** Let \( ABC \) be a triangle with area 1 (cm\(^2\)). Points \( D, E \) and \( F \) lie on the sides \( AB, BC \) and \( CA \), respectively. Prove that

\[
\min\{\text{Area of } \triangle ADF, \text{Area of } \triangle BED, \text{Area of } \triangle CEF\} \leq \frac{1}{4} \text{ (cm}^2)\]
Answer.

From the equalities
\[
\frac{S_{ADF}}{S_{ABC}} = \frac{AD \times AF}{AB \times AC}, \quad \frac{S_{BED}}{S_{ABC}} = \frac{BD \times BF}{AB \times AC}, \quad \frac{S_{CEF}}{S_{ABC}} = \frac{CE \times CF}{AB \times AC},
\]
we find
\[
\frac{S_{ADF}S_{BED}S_{CEF}}{(S_{ABC})^3} = \frac{(AD \times BD)(BE \times EC)(AF \times FC)}{AB^2 \times AC^2 \times BC^2}
\leq \frac{2}{2} \frac{2}{2} \frac{2}{2} = \frac{1}{64}.
\]
Hence,
\[
S_{ADF}S_{BED}S_{CEF} \leq \frac{1}{64} \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4}.
\]
It follows that
\[
\min\{\text{Area of } \triangle ADF, \text{Area of } \triangle BED, \text{Area of } \triangle CEF\} \leq \frac{1}{4}(\text{cm}^2).
\]

**Question 7.** Let \(ABC\) be a triangle with \(\hat{A} = 90^0, \hat{B} = 60^0\) and \(BC = 1\)cm. Draw outside of \(\triangle ABC\) three equilateral triangles \(ABD, ACE\) and \(BCF\). Determine the area of \(\triangle DEF\).

**Answer.**

From the assumption, we get \(AB = \frac{1}{2}, AC = \frac{\sqrt{3}}{2}\) and \(\overline{DBE} = 180^0\).

It is easy to check that
\[
S_{ABD} = \frac{1}{2}S_{ABC} = \frac{\sqrt{3}}{16}, \quad S_{BCF} = 2S_{ABC} = \frac{\sqrt{3}}{8}.
\]
Hence, \(S_{DEF} = \frac{\sqrt{3}}{16} + \frac{\sqrt{3}}{8} + \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{16} = \frac{9\sqrt{3}}{16}\)cm².

**Question 8.** Let \(ABCDE\) be a convex pentagon. Given that
\[
S_{\triangle ABC} = S_{\triangle BCD} = S_{\triangle CDE} = S_{\triangle DEA} = S_{\triangle EAB} = 2\text{cm}^2,
\]
Find the area of the pentagon.

**Answer.**

From the assumption
\[
S_{\triangle ABC} = S_{\triangle BCD} = S_{\triangle CDE} = S_{\triangle DEA} = S_{\triangle EAB} = 2\text{cm}^2,
\]
we find \(AB \parallel EC, BC \parallel AD, AC \parallel DE, AE \parallel BD\).
Let $O$ be the common point of $BD$ and $CE$. Denote $S_{BCO} = x$. Since $ABOE$ is a parallelogram, then $S_{ABE} = S_{BOE} = 2$ and

$$S_{ABCDE} = S_{ABE} + S_{BOE} + S_{CDE} + S_{BOC} = 6 + x.$$  

From

$$\frac{S_{BOC}}{S_{DOE}} = \frac{BO}{OD} = \frac{S_{BOE}}{S_{DOE}},$$

it follows $\frac{x}{2-x} = \frac{2}{x}$ since $S_{BOC} = S_{DOE}$, i.e. $x^2 + 2x + 1 = 5$ and then $x = \sqrt{5} - 1$. Hence

$$S_{ABCDE} = S_{ABE} + S_{BOE} + S_{CDE} + S_{BOC} = 6 + x = 6 + \sqrt{5} - 1 = 5 + \sqrt{5} \text{ cm}^2.$$  

Question 9. Solve the following system in positive numbers

$$\begin{cases} 
x + y \leq 1 \\
\frac{2}{xy} + \frac{1}{x^2 + y^2} = 10.
\end{cases}$$

Answer.

For every root $(x, y)$ of the system, we find

$$10 = \frac{2}{xy} + \frac{1}{x^2 + y^2} = \left(\frac{1}{2xy} + \frac{1}{x^2 + y^2}\right) + \frac{6}{4xy} \geq \frac{4}{(x + y)^2} + \frac{6}{(x + y)^2} \geq 4 + 6 = 10.$$  

Hence, the system is equivalent to

$$\begin{cases} 
x + y = 1 \\
x = y
\end{cases} \iff (x, y) = \left(\frac{1}{2}, \frac{1}{2}\right).$$

Question 10. Consider the set of all rectangles with a given perimeter $p$. Find the largest value of

$$M = \frac{S}{2S + p + 2},$$

where $S$ is denoted the area of the rectangle.

Answer.

Let $a, b$ be the lengths of sides of the rectangle, then $2(a + b) = p$, $ab = S$. By the Cauchy inequality, $p = 2(a + b) \geq 2 \times 2\sqrt{ab} = 4\sqrt{S}$. It follows $S \leq \frac{p^2}{16}$. Note that $0 < M < 1$, then

$$M = \frac{S}{2S + p + 2} \leq \frac{\frac{p^2}{16}}{\frac{p^2}{16} + p + 2} = \frac{p^2}{p^2 + 16p + 32}.$$  

The equality holds for $a = b$, i.e. $ABCD$ is a square.
Question 11. The positive numbers $a, b, c, d, e$ are such that the following identity hold for all real number $x$.

$$(x + a)(x + b)(x + c) = x^3 + 3dx^2 + 3x + e^3.$$ 

Find the smallest value of $d$.

**Answer.**

From the identity 

$$(x + a)(x + b)(x + c) = x^3 + 3dx^2 + 3x + e^3$$

we find

$$
\begin{cases}
    d = \frac{a + b + c}{3} \\
    ab + bc + ca = 3
\end{cases}
$$

Hence, by Cauchy inequality $(a+b+c)^2 \geq 3(ab+bc+ca)$, we get $d = \frac{a + b + c}{3} \geq \sqrt{\frac{ab + bc + ca}{3}} = 1$. The equality holds for $a = b = c = 1$.

**Question 12.** If $f(x) = ax^2 + bx + c$ satisfies the condition

$$|f(x)| < 1, \; \forall x \in [-1, 1],$$

prove that the equation $f(x) = 2x^2 - 1$ has two real roots.

**Answer.**

Rewrite the equation $f(x) = 2x^2 - 1$ in the form

$$g(x) := (2 - a)x^2 - bx - 1 - c = 0. \quad (1)$$

By the assumption,

$$
\begin{cases}
    f(-1) = a - b + c \\
    f(1) = a + b + c \\
    f(0) = c
\end{cases} \iff
\begin{cases}
    a = \frac{1}{2}[f(1) + f(-1)] - f(0) \\
    b = \frac{1}{2}[f(1) - f(-1)] \\
    c = f(0)
\end{cases}
$$

Hence, $|a| < 2$ and $|c| < 1$. These follow the equation (1) is a quadratic equation with $2 - a > 0$ and $-1 - c < 0$ then its discriminant $\Delta = b^2 - 4(2 - a)(-1 - c) > 0$, i.e. the equation (1) has real roots.

**Question 13.** Solve the system of equations

$$
\begin{cases}
    \frac{1}{x} + \frac{1}{y} = \frac{1}{6} \\
    \frac{3}{x} + \frac{2}{y} = \frac{5}{6}
\end{cases}
$$
Answer.

It is easy to check that
\[
\begin{align*}
\frac{1}{x} + \frac{1}{y} &= \frac{6}{5} \\
\frac{1}{x} + \frac{1}{y} &= \frac{6}{5} \\
\frac{1}{x} + \frac{1}{y} &= \frac{6}{5}
\end{align*}
\]
\[\Leftrightarrow \begin{align*}
\frac{3}{x} + \frac{3}{y} &= \frac{3}{1} \\
\frac{1}{x} + \frac{1}{y} &= \frac{1}{2} \\
\frac{1}{x} + \frac{1}{y} &= \frac{1}{2}
\end{align*} \Leftrightarrow (x, y) = (2, -3).
\]

Question 14. Solve the system of equations
\[
\begin{align*}
x^3 + y &= x^2 + 1 \\
2y^3 + z &= 2y^2 + 1 \\
3z^3 + x &= 3z^2 + 1
\end{align*}
\]

Answer.

Rewrite the system in the form
\[
\begin{align*}
x^2(x - 1) &= 1 - y \\
2y^2(y - 1) &= 1 - z \\
3z^2(z - 1) &= 1 - x
\end{align*}
\]

It follows that
\[
(x - 1)(y - 1)(z - 1)(6x^2y^2z^2 + 1) = 0. \tag{1}
\]

Since $6x^2y^2z^2 + 1 > 0$ for all $x, y, z$ then $(1) \Leftrightarrow x = 1$ or $y = 1$ or $z = 1$. For all cases, we always obtain the unique solution $(x, y, z) = (1, 1, 1)$.

Question 15. Denote by $\mathbb{Q}$ and $\mathbb{N}^*$ the set of all rational and positive integer numbers, respectively. Suppose that $\frac{ax + b}{x} \in \mathbb{Q}$ for every $x \in \mathbb{N}^*$. Prove that there exist integers $A, B, C$ such that
\[
\frac{ax + b}{x} = \frac{Ax + B}{Cx} \quad \text{for all } x \in \mathbb{N}^*.
\]

Answer.

Putting $x = 1, x = 2$ in $\frac{ax + b}{x}$ we get $a + b = p$, $\frac{2a + b}{2} = q$, where $p, q \in \mathbb{Q}$. So $a = 2q - p \in \mathbb{Q}$ and $b = 2p - 2q \in \mathbb{Q}$. Write $a = \frac{M}{N}$, $b = \frac{P}{Q}$, where $M, N, P, Q$ are integers. Hence
\[
\frac{ax + b}{x} = \frac{Mx + P}{Nx} = \frac{(MQ)x + (PN)}{(NQ)x},
\]
which was to be proved.