A SYNTHETIC PROOF OF DAO’S GENERALIZATION OF GOORMAGHTIGH’S THEOREM

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ABSTRACT. Using the concept of cross ratio, we give a synthetic proof of Dao’s generalization of Goormaghtigh’s theorem.

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1. INTRODUCTION

In 1930, René Goormaghtigh, French engineer and geometrician, expanded Droz-Farny theorem [1, 2, 3] with a nice theorem as follow.

Theorem 1.1 (Goormaghtigh [4]). Given triangle ABC and point P distinct from A, B, C. A line Δ passes through P. A1, B1, C1 belong to BC, CA, AB respectively such that PA1, PB1, PC1 are the images of PA, PB, PC respectively by reflection RΔ. Then, A1, B1, C1 are collinear.

Notation RΔ refers to reflection against Δ.

When P is the orthocenter of triangle ABC, theorem 1.1 actually becomes Droz-Farny theorem.

Proof of theorem 1.1 can be found in [5, 6].

In 2014, O.T.Dao expanded theorem 1.1 with two theorems [7].

In this article, we are first going to expand O.T.Dao’s second theorem with theorem 1.2 and more beautifully restate O.T.Dao’s first theorem with theorem 1.3. Then, we are going to prove theorems 1.2 and 1.3. Please note that, in terms of ideas, the way we prove theorems 1.2 and 1.3 is completely different from the way that theorem 1.1 is proved in [1] and [2].

Theorem 1.2. Given triangle ABC and point P distinct from A, B, C. A line Δ passes through P. α is any real number. Let A1, B1, C1 belong to BC, CA, AB respectively such that PA1, PB1, PC1 are the images of PA, PB, PC respectively by transformation RαP ◦ RΔ. Then, A1, B1, C1 are collinear.

Notation RαP refers rotation around P with angle of rotation α.

When α = 0, theorem 1.2 becomes theorem 1.1.

When α = π/2, theorem 1.2 becomes O.T.Dao’s second theorem.

Theorem 1.3 (Dao [7]). Given triangle ABC and point P distinct from A, B, C. Lines Δ and Δ’ cut at P. Points A1, B1, C1 belong to BC, CA, AB respectively such that (PA, PA1, Δ, Δ’) = (PB, PB1, Δ, Δ’) = (PC, PC1, Δ, Δ’) = −1. Then, A1, B1, C1 are collinear.
When $\Delta \perp \Delta'$, theorem 1.3 becomes theorem 1.1.

Theorem 1.3 is a different, more interesting reiteration of O.T.Dao's first theorem.

Before we prove theorems 1.2 and 1.3, note that notation $\overline{AB}$ refers to the signed length from point $A$ to point $B$.

2. PROOF OF THEOREM 1.2

There are two cases to consider.

Case 1. $\alpha \equiv 0 \pmod{2\pi}$. Then, $R_\alpha^p \circ R_\Delta = R_\Delta$.

Ignore platitudinous situations: $\Delta$ passes through a vertex of triangle $ABC$; $\Delta$ passes through two vertices of triangle of $ABC$.

Let $A_2, B_2$ be the intersections of $PC_1$ and $BC, CA$ respectively (see f.1).

![Figure 1](image)

Since reflection preserves cross ratio,

$$\frac{\overline{AB}}{\overline{AC}} : \frac{\overline{AB}}{\overline{AC}} = \frac{\overline{B_1 A_2}}{\overline{B_1 C_1}} = P(B_1 A_2) = P(BCA_1) = P(BCA_1) = P(B_1 C_1 A)$$

From this, noting that $A_2, B_2, C_1$ are collinear, by Menelaus theorem, we have

$$\frac{A_1 B}{A_1 C} : \frac{B_1 C}{B_1 A} = \frac{C_1 A}{C_1 A} = 1.$$

Hence, by Menelaus theorem, $A_1, B_1, C_1$ are collinear.

Case 2. $\alpha \not\equiv 0 \pmod{2\pi}$.

Let line $\Delta'$ pass through $P$ such that $\angle(\Delta, \Delta') \equiv \frac{\pi}{2} \pmod{\pi}$.

Apparently, $R_\alpha^p \circ R_\Delta = (R_\Delta \circ R_\alpha) \circ R_\Delta = R_\Delta \circ (R_\alpha \circ R_\Delta) = R_\Delta \circ id = R_\Delta$

From this, noting that $P$ belongs to $\Delta'$, according to case 1, we can deduce that $A_1, B_1, C_1$ are collinear.

3. PROOF OF THEOREM 1.3

We need two lemmas.

**Lemma 3.1.** If $BC, CA, AB$ are parallel to $B_1 C_1, C_1 A_1, A_1 B_1$ respectively, then two triangles $ABC$ and $A_1 B_1 C_1$ are similar in the same direction.
Proof. We have $BC / / B_1C_1$; $CA / / C_1A_1$; $AB / / A_1B_1$. Therefore, $\angle (BA, BC) \equiv \angle (B_1A_1, B_1C_1) \pmod {\pi}$ and $\angle (CA, CB) \equiv \angle (C_1A_1, C_1B_1) \pmod {\pi}$.

Hence, triangles $ABC$ and $A'B'C'$ are similar in the same direction.

Lemma 3.2. Given two triangles $ABC$ and $A_1B_1C_1$ which are similar in the same direction. $A_2, B_2, C_2$ are the midpoints of $AA_1, BB_1, CC_1$ respectively. Then, triangle $A_2B_2C_2$ are similar to triangles $ABC$ and $A_1B_1C_1$ in the same direction.

Proof. Let $M, N$ be the midpoints of $AB_1, AC_1$ respectively (see f.2).

Because $M, N, A_2$ are the midpoints of $AB_1, AC_1, AA_1$ respectively, $MN, NA_2, A_2M$ are parallel to $B_1C_1, C_1A_1, A_1B_1$ respectively. Therefore, by lemma 3.1, triangles $A_2MN$ and $A_1B_1C_1$ are similar in the same direction (1).

As $A_2, B_2, C_2, M, N$ are the midpoints of $AA_1, BB_1, CC_1, AB_1, AC_1$ respectively, $\overrightarrow{MA_2} = \frac{1}{2} \overrightarrow{B_1A_1}; \overrightarrow{MB_2} = \frac{1}{2} \overrightarrow{AB}; \overrightarrow{NA_2} = \frac{1}{2} \overrightarrow{C_1A_1}; \overrightarrow{NC_2} = \overrightarrow{AC}$.

![Figure 2](image)

From this, noting that triangles $ABC$ and $A_1B_1C_1$ are similar in the same direction,

$$\angle (\overrightarrow{MA_2}, \overrightarrow{MB_2}) \equiv \angle (\overrightarrow{B_1A_1}, \overrightarrow{AB}) \equiv \pi + \angle (\overrightarrow{B_1A_1}, \overrightarrow{BA}) \pmod {2\pi}$$

$$\equiv \pi + \angle (\overrightarrow{C_1A_1}, \overrightarrow{CA}) \equiv \angle (\overrightarrow{C_1A_1}, \overrightarrow{AC}) \equiv \angle (\overrightarrow{NA_2}, \overrightarrow{NC_2}) \pmod {2\pi}.$$ 

Thus, triangles $A_2MB_2$ and $A_2NC_2$ are similar in the same direction. Therefore, triangles $A_2MN$ and $A_2B_2C_2$ are similar in the same direction (2).

From (1) and (2), deduce that $A_1B_1C_1$ and $A_2B_2C_2$ are similar in the same direction. In other words, triangle $A_2B_2C_2$ are similar to triangles $ABC$ and $A_1B_1C_1$ in the same direction.

Return to the proof of theorem 1.3.

Let $A_2, B_2$ be the intersections of $PC_1$ and $BC, CA$ respectively. Let $A_3, B_3, C_3$ be the intersections of $PA_1, PB_1, PC_1$ and the lines parallel to $\Delta$, passing through $A, B, C$ respectively. Let $A_0, B_0, C_0$ be the intersections of $\Delta$ and $AA_3, BB_3, CC_3$ respectively (see f.3).
Because \((PA, PA_1, \Delta, \Delta') = (PB, PB_1, \Delta, \Delta') = (PC, PC_1, \Delta, \Delta') = -1\),
\((PA, PA_3, PA_0, \Delta) = (PB, PB_3, PB_0, \Delta) = (PC, PC_3, PC_0, \Delta) = -1\).
Then, combined with the fact that \(AA_3, BB_3, CC_3\) are all parallel to \(\Delta\), we can deduce that \(A_0, B_0, C_0\) are the midpoints of \(AA_3, BB_3, CC_3\) respectively.
If \(BC, CA, AB\) are parallel to \(B_3C_3, C_3A_3, A_3B_3\) respectively, then by lemma 3.1, triangles \(ABC\) and \(A_3B_3C_3\) are similar in the same direction. From this, noting that \(A_0, B_0, C_0\) are the midpoints of \(AA_3, BB_3, CC_3\) respectively, by lemma 3.2, we can deduce that \(A_0, B_0, C_0\) are not collinear, contradiction. Thus, \(BC, CA, AB\) are not respectively parallel to \(B_3C_3, C_3A_3, A_3B_3\). Without the loss of generality, assume that \(BC\) and \(B_3C_3\) are not parallel.
Let \(S\) be the intersection of \(BC\) and \(B_3C_3\). Let \(A_2, A_4\) be the intersections of \(BC\) and \(C_1P, AA_3\) respectively. Let \(A_5, A_6, A_7\) be the intersections of \(B_3C_3\) and \(AP, CP, AA_3\) respectively.
Applying Ceva’s theorem to triangle \(SC_3C\), noting that \(SC_0, C_3A_2, CA_6\) are concurrent (at \(P\)), we have
\[
\frac{C_0C_3}{C_0C} \cdot \frac{A_2C}{A_2S} \cdot \frac{A_6S}{A_6C_3} = -1.
\]
Combined with the fact that \(C_0\) is the midpoint of \(C_3C\), we have \(\frac{A_6S}{A_6C} = \frac{A_7S}{A_7C_3}\).
Therefore, by Thales theorem, \(A_2A_6 // CC_3\) (3).
Applying Menelaus theorem to triangles $A_0SA_4$ and $A_0SA_7$, noting that $A_1, A_3, P$ are collinear and $A_5, A, P$ are collinear, we have
\[
\frac{A_1S}{A_1A_4} \cdot \frac{A_3A_4}{A_3A_0} \cdot \frac{PA_0}{PS} = 1 = \frac{A_5S}{A_5A_7} \cdot \frac{AA_7}{AA_0} \cdot \frac{PA_0}{PS}.
\]
From this, noting that $A_0$ is the midpoint of both $AA_3$ and $A_4A_7$, deduce that
\[
\frac{A_1S}{A_1A_4} = \frac{A_5S}{A_5A_7}.
\]
Therefore, by Thales theorem, $A_1A_5 // A_4A_7$ (4).
From (3) and (4), deduce that $BB_3 // CC_3 // A_1A_5 // A_2A_6$.
Hence,
\[
\frac{A_1B}{A_1C} : \frac{A_2B}{A_2C} = \frac{BCA_1A_2}{BCA_3A_5A_6} = \frac{B(B_1B_2AC)}{B(B_3C_3A_5A_6)} = P(B_1B_2AC) = P(ACB_1B_2) = \frac{B_1C}{B_2C} \cdot \frac{B_2A}{B_3A}.
\]
From this, noting that $A_2, B_2, C_1$ are collinear, by Menelaus theorem, deduce that
\[
\frac{A_1B}{A_1C} \cdot \frac{B_1C}{B_1A} \cdot \frac{C_1A}{C_1B} = \frac{A_2B}{A_2C} \cdot \frac{B_2C}{B_2A} \cdot \frac{C_1A}{C_1B} = 1.
\]
Thus, by Menelaus theorem, $A_1, B_1, C_1$ are collinear.

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REFERENCES


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