AN INTERESTING APPLICATION OF THE BRITISH FLAG THEOREM

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ABSTRACT. We will use the British flag theorem to prove an elegant theorem for two similarly oriented regular polygons-2n.

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1. INTRODUCTION

The British flag theorem is one of the simplest theorems in plane geometry.

Theorem 1.1 (British flag). If ABCD be a rectangle and P be any point on the plane, then

\[ PA^2 + PC^2 = PB^2 + PD^2 \]  

(1)

Theorem 1.1 could easily be given as an assignment for secondary school students after they have learnt the Pythagoras theorem. Theorem 1.1 can be found in [1,p.87]. It is impossible to list all the applications of theorem 1.1. In this article, by proving a new theorem, an elegant theorem for two similarly oriented regular polygons-2n, we will be introducing another interesting application of theorem 1.1.

The new theorem is stated using the concept of signed area of a quadrilateral.

Definition 1.1. The signed area of a quadrangle XYZT is a number, denoted as \( S[XYZT] \), and defined as \( S[XYZT] = \frac{1}{2} \langle XZ \land YT \rangle \), where notation \( a \land b \) refers to the cross product of two vectors \( a \) and \( b \), i.e. \( a \land b = \frac{1}{2} |a| |b| \sin(a,b) \), where \( (a,b) \) is the directional angle between two vectors \( a \) and \( b \).


Denote the area of a polygon as \( S(\cdot) \).

- \( S[XYZT] = S(XYZT) \) if quadrangle XYZT is convex and positively orientated (f.1a);
- \( S[XYZT] = S(XYZ) - S(XTZ) \) if quadrangle XYZT is concave at T and triangle XYZ is positively orientated (f.1b);
- \( S[XYZT] = S(XYO) - S(ZTO) \) if quadrangle XYZT cuts itself at \( O = XT \cap YZ \) and triangle XYO is positively orientated (f.1c);
- \( S[XYZT] = S(ZTO) - S(XYO) \) if quadrangle XYZT cuts itself at \( O = XT \cap YZ \) and triangle XYO is negatively orientated (f.1d).

The yellow triangles on figures 1 are positively orientated (1.a, 1.b, 1.c, 1.d) and the green ones are negatively orientated (1.b, 1.c, 1.d). Definition 1.1 can be found in [2, pp. 178-184].
Theorem 1.2. If $A_1A_2...A_{2n}$ and $B_1B_2...B_{2n}$ are two similarly oriented regular polygons, then $S[A_iA_{i+1}B_{i+1}B_i] + S[A_{n+i}A_{n+i+1}B_{n+i+1}B_{n+i}]$ is constant for any $i \in \{1; 2; \ldots; 2n\}$, assuming that $A_{2n+1} = A_1$ and $B_{2n+1} = B_1$.

Due to the concept of signed area in theorem 1.2, regular polygon $B_1B_2...B_{2n}$ does not have to lie inside regular polygon $A_1A_2...A_{2n}$; quadrangles $A_iA_{i+1}B_{i+1}B_i$ and $A_{n+i}A_{n+i+1}B_{n+i+1}B_{n+i}$ can cut themselves for any $i \in \{1; 2; \ldots; 2n\}$, assuming that $A_{2n+1} = A_1$ and $B_{2n+1} = B_1$.

2. PROOF OF THEOREM 1.2

First, we need one lemma.

Lemma 2.1. If $ABCD$ and $A_0B_0C_0D_0$ are two similar and similarly oriented rectangles, then

$$S[ABB_0A_0] + S[CDD_0C_0] = \frac{1}{2} (AB \wedge AC - A_0B_0 \wedge A_0C_0).$$

Proof of Lemma 2.1. Because $ABCD$ and $A_0B_0C_0D_0$ are similar and similarly oriented, there exist a point $P$, which is the centre of spiral similarity transforming $ABCD$ into $A_0B_0C_0D_0$ and real numbers $k$ and $\alpha$ such that (f.2).

$$\frac{PA}{P_0A} = \frac{PB}{P_0B} = \frac{PC}{P_0C} = \frac{PD}{P_0D} = k;$$

$$(PA, PA_0) \equiv (PB, PB_0) \equiv (PC, PC_0) \equiv (PD, PD_0) \equiv \alpha \pmod{2\pi}).$$

Thus, by theorem 1.1, noting that $CD = -AB; C_0D_0 = -A_0B_0$, we have
\[ 2 \left( S[A B B_0 A_0] + S[C D D_0 C_0] \right) \]
\[ = A B_0 \land B A_0 + C D_0 \land D C_0 \]
\[ = (P B_0 - P A) \land (P A_0 - P B) + (P D_0 - P C) \land (P C_0 - P D) \]
\[ = - P B_0 \land P B - P A \land P A_0 + P A \land P B + P B_0 \land P A_0 \]
\[ - P D_0 \land P D - P C \land P C_0 + P C \land P D + P D_0 \land P C_0 \]
\[ = P B_0 \cdot P B \sin \alpha - P A \cdot P A_0 \sin \alpha + P D_0 \cdot P D \sin \alpha - P C \cdot P C_0 \sin \alpha \]
\[ + P A \land (P A + A B) + P C \land (P C + C D) + (P A_0 + A_0 B_0) \land P A_0 + (P C_0 + C_0 D_0) \land P C_0 \]
\[ = k \sin \alpha \left( P B^2 + P D^2 - P A^2 - P C^2 \right) + P A \land A B + P C \land C D + A_0 B_0 \land P A_0 + C_0 D_0 \land P C_0 \]
\[ = - A B \land P A + A B \land P C + A_0 B_0 \land P A_0 - A_0 B_0 \land P C_0 \]
\[ = A B \land (P C - P A) - A_0 B_0 \land (P C_0 - P A_0) \]
\[ = (A B \land A C - A_0 B_0 \land A_0 C_0) . \]

Therefore, \( S[A B B_0 A_0] + S[C D D_0 C_0] = \frac{1}{2} (A B \land A C - A_0 B_0 \land A_0 C_0) . \quad \square \)

**Note.** A Spiral similarity with center \( P \), rotation angle \( \alpha \) and similarity coefficient \( k \) is the sum of a central similarity with center \( P \) and similarity coefficient \( k \) and a rotation about \( P \) through the angle \( \alpha \), taken in either order [3, p.36].

Next, we are going to prove theorem 1.2 (f.3.a, f.3.b).

Without the loss of generality, assume that \( A_1 A_2 \ldots A_{2n} \) and \( B_1 B_2 \ldots B_{2n} \) are positively oriented.

Let \( O_a \) and \( O_b \) are the centres of \( A_1 A_2 \ldots A_{2n} \) and \( B_1 B_2 \ldots B_{2n} \) respectively.

Because \( A_1 A_2 \ldots A_{2n} \) and \( B_1 B_2 \ldots B_{2n} \) are regular polygons that share a positive orientation, \( A_i A_{i+1} A_{i+1+n} A_{i+n} \) and \( B_i B_{i+1} B_{i+1+n} B_{i+n} \) are similar and positively oriented rectangles for any \( i \in \{1; 2; \ldots; n\} \), assuming that \( A_{2n+1} = A_1 \) and \( B_{2n+1} = B_1 \).

Hence, by the lemma 2.1, we have

\[ S[A_i A_{i+1} B_{i+1} B_i] + S[A_{i+n} A_{i+1+n} B_{i+1+n} B_i] \]
\[ = \frac{1}{2} (A_i A_{i+1} \land A_i A_{i+n} - B_i B_{i+1} \land B_i B_{i+n}) \]
\[ = \frac{1}{2} (A_i A_{i+1} A_i A_{i+n} \sin (A_i A_{i+1}, A_i A_{i+n}) - B_i B_{i+1} B_i B_{i+n} \sin (B_i B_{i+1}, B_i B_{i+n})) \]
\[ = \frac{1}{2} \left( A_i A_{i+1} A_i A_{i+n} \sin A_{i+1} A_i A_{i+n} - B_i B_{i+1} B_i B_{i+n} \sin B_{i+1} B_{i+n} \right) \]
\[ = \frac{1}{2} \left( 2S(A_i A_{i+1} A_{i+n}) - 2S(B_i B_{i+1} B_{i+n}) \right) \]
\[ = \frac{1}{2} \left( 4S(O_a A_i A_{i+1}) - 4S(O_b B_i B_{i+1}) \right) \]
\[ = 2 \left( \frac{1}{2} \frac{1}{n} S(A_1 A_{2} \ldots A_{2n}) - \frac{1}{2} \frac{1}{n} S(B_1 B_{2} \ldots B_{2n}) \right) \]
\[ = \frac{1}{n} \left( S(A_1 A_{2} \ldots A_{2n}) - S(B_1 B_{2} \ldots B_{2n}) \right) . \]
This means that \( S [A_i A_{i+1} B_{i+1} B_i] + S [A_{n+i+1} A_{n+i+1} B_{n+i+1} B_{n+i}] \) is constant for any \( i \in \{1; 2; \ldots; 2n\} \), assuming that \( A_{2n+1} = A_1 \) and \( B_{2n+1} = B_1 \).

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