Transcendence of e by Rich Schwartz

I adapted this proof from the one in §5.2 of Herstein's *Topics in Algebra*. I think this proof is simpler and more businesslike.

The Main Step: Assume e is algebraic. Then e satisfies a polynomial equation with integer coefficients, having the following form.

$$\sum_{k=0}^{n} c_k e^k = 0; \qquad c_0 \neq 0; \qquad \max_k |c_k| < n. \tag{1}$$

Note that the degree of this equation might be less than n.

Below, we will produce an integer $p > n$ and a list $F(0), ..., F(n)$ of integers such that

1. $F(0) \in \boldsymbol{Z} - p\boldsymbol{Z}$.
2. $F(1), ..., F(n) \in p\boldsymbol{Z}$.
3. $|F(k) - e^k F(0)| < 1/n^2$ for $k = 1, ..., n$.

We have

$$1 \leq^* \Big|\sum_{k=0}^{n} c_k F(k)\Big| = \Big|\sum_{k=0}^{n} c_k F(k) - 0 \times F(0)\Big| = \Big|\sum_{k=0}^{n} c_k F(k) - \Big(\sum_{k=0}^{n} c_k e^k\Big) \times F(0)\Big|$$

$$= \Big|\sum_{k=0}^{n} c_k \Big(F(k) - e^k F(0)\Big)\Big| < n \sum_{k=0}^{n} |F(k) - e^k (F(0)| < 1. \tag{2}$$

The starred inequality needs explanation. Since $0 < |c_0| < n$, we have $c_0 F(0) \in \boldsymbol{Z} - p\boldsymbol{Z}$. Also, $c_k F(k) \in p\boldsymbol{Z}$ for all $k = 1, ..., n$. So, the right hand side of the starred inequality lies in $\boldsymbol{Z} - p\boldsymbol{Z}$ and hence is a nonzero integer. The contradiction is that $1 < 1$. Hence e is transcendental.

Producing the List of Integers: It remains to produce the magic list of integers. Consider the function

$$F = \sum_{i=0}^{\infty} f^{(i)}; \qquad f(x) = \frac{x^{p-1}(1-x)^p(2-x)^p...(n-x)^p}{(p-1)!}; \tag{3}$$

Here $f^{(i)}$ is the ith derivative of f. The sum for F is finite, because f is a polynomial. f is called a *Hermite polynomial*.

Property 1: We can write $f = a \times b$, where

$$a(x) = \frac{x^{p-1}}{(p-1)!}; \qquad b(x) = (1-x)^p ... (n-x)^p. \tag{4}$$

By the product rule for derivatives,

$$f^{(N)} = \sum_{i=0}^{N} a^{(i)} b^{(N-i)}. \tag{5}$$

We have $a^{(p-1)}(0) = 1$ and otherwise $a^{(i)}(0) = 0$. Hence

$$F(0) = \sum_{i=0}^{\infty} b^{(i)}(0) = b(0) + \sum_{i=1}^{\infty} b^{(i)}(0) = (n!)^p + p(...) \in \boldsymbol{Z} - p\boldsymbol{Z}.$$

Property 2: We can write $f = a \times b$, where

$$a(x) = \frac{(x-k)^p}{(p-1)!}; \qquad b(x) = \frac{x^{p-1}(1-x)^p ... (n-x)^p}{(k-x)^p}. \tag{6}$$

Note that $b \in \boldsymbol{Z}[x]$. We again have Equation 5. This time $a^{(p)}(k) = p$ and otherwise $a^{(i)}(k) = 0$. Hence

$$F(k) = p \times \sum_{i=0}^{\infty} b^{(i)}(k) \in p\boldsymbol{Z}.$$

Property 3: Let $\phi(x) = e^{-x} F(x)$. We compute

$$\phi'(x) = -e^{-x}(F(x) - F'(x)) = -e^{-x}\Big(\sum_{i=0}^{\infty} f^{(i)}(x) - \sum_{i=1}^{\infty} f^{(i)}(x)\Big) = -e^{-x} f(x).$$

The sums are finite, because f is a polynomial. Our equation tells us that $|\phi'(x)| \leq |f(x)|$ for $x \geq 0$. Hence

$$|F(k) - e^k F(0)| = |e^k||\phi(k) - \phi(0)| \leq k e^k \max_{[0,k]} |\phi'| \leq n e^n \max_{[0,n]} |f| \leq \frac{e^n (n^{n+2})^p}{(p-1)!}$$

For p sufficiently large, this last bound is less than $1/n^2$.