Problem 1. *Link*

Find all functions $f : R \longrightarrow R$ such as:

$$f(x^2+y^2) = f(x^2) + f(y^2) + 2f(x)f(y)$$

Bài Toán 1. *Link*

Tìm các hàm số $f : R \longrightarrow R$ thỏa mãn:

$$f(x^2+y^2) = f(x^2) + f(y^2) + 2f(x)f(y)$$

Solution.

Let $P(x,y)$ the assertion

$$f(x^2+y^2) = f(x^2) + f(y^2) + 2f(x)f(y)$$

$$P(x,0) \implies f(0)(1+2f(x)) = 0$$

and so, if $f(0) \neq 0$, then $f(x) = -\dfrac{1}{2} \quad \forall x$ which indeed is a solution.

Let us from now consider that $f(0) = 0$

$f(x) = 0 \quad \forall x$ is also a solution and let us from now consider that $\exists u$ such that $f(u) \neq 0$

Comparing $P(u,x)$ and $P(u,-x)$, we get $f(-x) = f(x)$ and $f(x)$ is an even function and we can write $f(x) = g(x^2)$

Notice then that $P(x,y)$ becomes

$$P'(x,y) : g(x^2+2xy+y^2) = g(x^2) + g(y^2) + 2g(x)g(y) \forall x, y \geq 0$$

1) $\exists h(x)$ such that we have new assertion

$$Q(x,y) : g(x+y) = g(x)h(y) + g(y) \forall x, y \geq 0$$

$$\begin{aligned} P(x, \sqrt{y^2+z^2}) \implies f(x^2+y^2+z^2) &= f(x^2) + f(y^2+z^2) + 2f(x)f(\sqrt{y^2+z^2}) \\ &= f(x^2) + f(y^2) + f(z^2) + 2f(x)f(\sqrt{y^2+z^2}) + 2f(y)f(z) \end{aligned}$$

Same, :

$$P(y, \sqrt{x^2+z^2}) \implies f(x^2+y^2+z^2) = f(x^2) + f(y^2) + f(z^2) + 2f(y)f(\sqrt{x^2+z^2}) + 2f(x)f(z)$$

And so

$$f(x)f(\sqrt{y^2+z^2}) + f(y)f(z) = f(y)f(\sqrt{x^2+z^2}) + f(x)f(z)$$

Which implies

$$g(x^2)g(y^2+z^2) + g(y^2)g(z^2) = g(y^2)g(x^2+z^2) + g(x^2)g(z^2)$$

And so

$$\begin{gathered} g(x)g(y+z) + g(y)g(z) = g(y)g(x+z) + g(x)g(z) \forall x, y, z \geq 0 \\ \implies g(x)(g(y+z) - g(z)) = g(y)(g(x+z) - g(z)) \end{gathered}$$

Let $y = u^2$ such that $g(y) = f(u) \neq 0$. the above equation becomes

$$g(x+z) - g(z) = g(x)\frac{g(u^2+z) - g(z)}{g(u^2)}$$

And so $g(x+z) - g(z) = g(x)h(z)$ for some function $h(x)$ Q.E.D.
2) The only possibilities are $h(x) = 0$ and $h(x) = 1$

$$Q(x, u^2) \implies g(x+u^2) = g(x)h(u^2) + g(u^2)$$

$$Q(y^2, x) \implies g(x+u^2) = g(u^2)h(x) + g(x)$$

Subtracting, we get $g(u^2)(h(x)-1) = g(x)(h(u^2)-1)$
If $h(u^2) = 1$, we get $h(x) = 1 \quad \forall x$ and so $g(x+y) = g(x) + g(y)$
Then $P'(x, y)$ implies $g(xy) = g(x)g(y)$
And this double equation is very classical and has a unique non constant solution $g(x) = x$ and so $f(x) = x^2$ which indeed is a solution
If $h(u^2) = \neq 1$, we get $g(x) = a(h(x)-1)$ But then :

$$Q(u^2, x+y) \implies g(u^2+x+y) = g(u^2)h(x+y) + g(x+y) = g(u^2)h(x+y) + g(x)h(y) + g(y)$$

$$Q(u^2+x, y) \implies g(u^2+x+y) = g(u^2+x)h(y) + g(y) = g(u^2)h(x)h(y) + g(x)h(y) + g(y)$$

And so $h(x+y) = h(x)h(y) \quad \forall x, y \geq 0$
This is a classical equation which gives : either $h(x) = 0 \quad \forall x$, either $h(x) = e^{c(x)}$ where $c(x)$ is any solution of Cauchy equation.
$h(x) = 0 \quad \forall x \implies g(x)$ is constant and so $f(x)$ is constant and so the two solutions we already know $f(x) = 0$ or $f(x) = -\dfrac{1}{2}$
$h(x) = e^{c(x)}$: we get $g(x) = a(e^{c(x)} - 1)$ Plugging this in $P'(x, y) \implies g(x) = 0$ and so $f(x) = 0$ Notice that this point is not obvious at all and I had a lot of difficulties to show it. I could give the directions I used if somebody asked.
3) Synthesis of solutions :
We found three solutions : $f(x) = 0 \quad \forall x$; $f(x) = -\frac{1}{2} \quad \forall x$; $f(x) = x^2 \quad \forall x$. ■

Problem 2. ***Link***
Solve this inequation for $x \in R$

$$\sqrt{x^2 - x - 2} + 3\sqrt{x} \leq \sqrt{5x^2 - 4x - 6}$$

Bài Toán 2. ***Link***
Giải bất phương trình sau với $x \in R$

$$\sqrt{x^2 - x - 2} + 3\sqrt{x} \leq \sqrt{5x^2 - 4x - 6}$$

Solution.
$\iff \; (x+1)(x-2) \geq 0$ and $x \geq 0$ and $5x^2 - 4x - 6 \geq 0$
and (squaring) $3\sqrt{x^3 - x^2 - 2x} \leq 2x^2 - 6x - 2$
$\iff \; x \geq 2$ and $2x^2 - 6x - 2 \geq 0$ and (squaring) $4x^4 - 33x^3 + 37x^2 + 42x + 4 \geq 0$
$\iff \; x \geq \frac{3+\sqrt{13}}{2}$ and $(x^2 - 6x - 4)(4x^2 - 9x - 1) \geq 0 \iff x \geq 3 + \sqrt{13}$ ■

Problem 3. ***Link***

$0 < x_1 < 2$, $x_{n+1} = x_n(x_n - 1)$. Find $\lim x_n$

Bài Toán 3. ***Link***

Cho $0 < x_1 < 2$, $x_{n+1} = x_n(x_n - 1)$.Tìm $\lim x_n$

Solution.

We get $x_n \in [-\frac{1}{4}, 2)$ and $x_{n+2} = x_n^4 - 2x_n^3 + x_n = x_n g(x_n)$ with $g(x) = x^3 - 2x^2 + 1$
If $x_k = 0$ for some k, then $x_n = 0$ $\forall n \geq k$
If $x_n \neq 0$ $\forall n$, it's easy to show that for $x \in [-\frac{1}{4}, 0) \cup (0, 2)$: $|g(x)| < 1$ and so $|x_{n+2}| < |x_n|$
So $|x_{2n}|$ is a positive decreasing sequence. So it has a limit $\in [0, 2)$, which is root of $x^4 - 2x^2 + x = x$, and so is zero. Same for x_{2n+1}
Hence the answer : $\boxed{\lim\limits_{n\to+\infty} x_n = 0}$ ■

Problem 4. ***Link***

Solve the system of equation:

$$\begin{cases} x^4 - y^4 = \dfrac{121x - 122y}{4xy} \\ x^4 + 14x^2y^2 + y^4 = \dfrac{122x + 121y}{x^2 + y^2} \end{cases}$$

Bài Toán 4. ***Link***

Giải hệ phương trình:

$$\begin{cases} x^4 - y^4 = \dfrac{121x - 122y}{4xy} \\ x^4 + 14x^2y^2 + y^4 = \dfrac{122x + 121y}{x^2 + y^2} \end{cases}$$

Solution.

■

Problem 5. *Link*

Solve in integers the following equation

$$(xy-1)^2 = (x+1)^2 + (y+1)^2$$

Bài Toán 5. *Link*

Giải trong tập số nguyên phương trình sau

$$(xy-1)^2 = (x+1)^2 + (y+1)^2$$

Solution.

Firstly, we will show that if $|x|, |y| > 2$ then there is no solution.

i) $x, y > 0$. In this case by symmetry, we can assume that $x \geq y$

Then $xy-1 \geq 3x-1 > \sqrt{2}(x+1)$ since $x, y > 2$ and so $(xy-1)^2 > 2(x+1)^2 \geq (x+1)^2+(y+1)^2$ which is a contradiction.

ii) $x, y < 0$. In this case let $x = -a,\ y = -b$ where $a, b > 2$ and $a \leq b$

Hence $ab+1 \geq 3b+1 > \sqrt{2}(b+1) > \sqrt{2}(b-1)$. Hence $(ab+1)^2 > 2(b-1)^2 \geq (a-1)^2+(b-1)^2$ which is a contradiction.

iii) $xy < 0$. In this case since $x \geq y,\ x > 0,\ y < 0$. Let $y = -z,\ x, z > 2$

$(xz+1)^2 = (x+1)^2 + (z-1)^2$

We can show that $(xz+1)^2 > (x+1)^2 + (z+1)^2$ when $x, z > 2$ in a similar way in part **i)**

So, the only remaining case is $\min\{|x|, |y|\} \leq 2$

$x = 0 \implies y = -1$

$x = 1 \implies y = -1$

$x = -1 \implies$ the equation holds for all $y \in \mathbb{Z}$

$x = 2 \implies y = -1 \lor y = 3$

$x = -2 \implies y = -1$

So, all solutions are $\{(2,3),(3,2)\} \cup \{(x,y) \in \mathbb{Z}\times\mathbb{Z} : x = -1 \lor y = -1\}$

Another way: Rearrange to $(x+1)(y+1)((x-1)(y-1)-2) = 0$ ■