Combinatorial Identities: Table I: Intermediate Techniques for Summing Finite Series

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1 Coefficient Comparison

Remark 1.1 *Throughout this chapter, we will assume, unless otherwise specified, that x is an arbitrary complex number and n is a nonnegative integer.*

1.1 Expansions of $(1+x)^n(a+x)^n$

Remark 1.2 *In this section, we will assume a is an arbitrary complex number.*

$$\sum_{k=0}^{n}\binom{n}{k}\binom{2n-k}{n}(a-1)^k = \sum_{k=0}^{n}\binom{n}{k}^2 a^{n-k} \tag{1.1}$$

1.1.1 Equation (1.1) with $a = 0$

$$\sum_{k=0}^{n}\binom{n}{k}\binom{2n-k}{n}(-1)^k = 1 \tag{1.2}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n+k}{n} = (-1)^n \tag{1.3}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n+k}{n-k}\binom{2k}{k} = (-1)^n \tag{1.4}$$

$$\sum_{k=1}^{n}(-1)^{n-k}\binom{n+k-1}{k}\binom{n-1}{k-1} = 1, \qquad n \geq 1 \tag{1.5}$$

1.1.2 Equation (1.1) with $a = 1$

Special Case of the Vandermonde Convolution

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n} \tag{1.6}$$

1.1.3 Equation (1.1) with $a = -1$

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{4n-k}{2n} 2^k = (-1)^n \binom{2n}{n} \tag{1.7}$$

1.1.4 Equation (1.1) with $a = 2$

Remark 1.3 *The following Identity, due to Leo Moser, is Problem E799 of The American Math. Monthly, 1948, P. 30.*

$$\sum_{k=0}^{n} \binom{n}{k} \binom{2n-k}{n} = \sum_{k=0}^{n} \binom{n}{k}^2 2^{n-k} \tag{1.8}$$

1.1.5 Shifted Version of Equation (1.1)

$$\sum_{k=0}^{n+1} \binom{n}{k} \binom{2n-k}{n+1-k} (a-1)^k = \sum_{k=1}^{n} \binom{n}{k} \binom{n}{n+1-k} a^{n+1-k}, \qquad n \geq 1 \tag{1.9}$$

Equation (1.9) with $a = 1$

$$\sum_{k=1}^{n} \binom{n}{k} \binom{n}{n+1-k} = \binom{2n}{n+1}, \qquad n \geq 1 \tag{1.10}$$

Equation (1.9) with $a = 0$

$$\sum_{k=0}^{n+1} (-1)^k \binom{n}{k} \binom{2n-k}{n+1-k} = 0 \tag{1.11}$$

1.2 Expansions of $(\alpha x+\beta)^p(1+x)^q$

Remark 1.4 *In this subsection, we assume α and β are arbitrary real or complex numbers. We note that q is an arbitrary real number, while p is a nonnegative integer.*

Remark 1.5 *The following identity is found in Wilhelm Ljunggren's "Et elemnært bevis for en formel av A.C. Dixon", Norsk Matematisk Tidsskrift, 29.Ågang, 1947, pp. 35-38.*

$$\boldsymbol{\sum_{k=0}^{p}\binom{p}{k}\binom{q}{k}\alpha^{p-k}\beta^{k}=\sum_{k=0}^{p}\binom{p}{k}\binom{q+k}{k}(\alpha-\beta)^{p-k}\beta^{k}} \tag{1.12}$$

1.2.1 Dixon Sums

Dixon's Identity

$$\boldsymbol{\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^3=(-1)^n\binom{2n}{n}\binom{3n}{n}} \tag{1.13}$$

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^3=\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{4n-k}{2n}\binom{2n+k}{2n} \tag{1.14}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2n-k}{n}\binom{n+k}{n} \tag{1.15}$$

1.3 Expansions of $(1+x)^{2n}(1-x)^{2n}$

$$\boldsymbol{\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^2=(-1)^n\binom{2n}{n}} \tag{1.16}$$

$$\sum_{k=0}^{n}\binom{2n}{2k}^2=\frac{1}{2}\binom{4n}{2n}+\frac{(-1)^n}{2}\binom{2n}{n} \tag{1.17}$$

$$\sum_{k=0}^{n-1}\binom{2n}{2k+1}^2=\frac{1}{2}\binom{4n}{2n}-\frac{(-1)^n}{2}\binom{2n}{n}, \qquad n\geq 1 \tag{1.18}$$

1.4 Expansions of $(1-x)^n(1+x)^n$

Remark 1.6 *In this section, we assume r is a nonnegative integer.*

$$\sum_{k=0}^{r}(-1)^k\binom{n}{k}\binom{n}{r-k} = \begin{cases} 0, & \text{if } r \text{ is odd} \\ (-1)^{\frac{r}{2}}\binom{n}{\frac{r}{2}}, & \text{if } r \text{ is even} \end{cases} \tag{1.19}$$

1.4.1 Restatements of Equation (1.19)

First Restatement of Equation (1.19)

$$\sum_{k=0}^{2r+1}(-1)^k\binom{n}{k}\binom{n}{2r-k+1} = 0 \tag{1.20}$$

$$\sum_{k=0}^{2r}(-1)^k\binom{n}{k}\binom{n}{2r-k} = (-1)^r\binom{n}{r} \tag{1.21}$$

Second Restatement of Equation (1.19)

$$\sum_{k=0}^{r}(-1)^k\binom{n}{k}\binom{n}{r-k} = (-1)^{[\frac{r}{2}]}\binom{n}{[\frac{r}{2}]}\frac{1+(-1)^r}{2}, \tag{1.22}$$

where, for any real number x, we let $[x]$ denote the greatest integer in x.

1.4.2 Applications of Equation (1.21)

$$\sum_{k=0}^{2r}(-1)^k\binom{2r}{k}\binom{2n-2r}{n-k} = (-1)^r\frac{\binom{n}{r}\binom{2n}{n}}{\binom{2n}{2r}} \tag{1.23}$$

$$\sum_{k=0}^{r}\binom{n}{2k}\binom{n}{2r-2k} = \frac{1}{2}\binom{2n}{2r} + \frac{(-1)^r}{2}\binom{n}{r} \tag{1.24}$$

$$\sum_{k=0}^{r-1}\binom{n}{2k+1}\binom{n}{2r-2k-1} = \frac{1}{2}\binom{2n}{2r} - \frac{(-1)^r}{2}\binom{n}{r}, \qquad r \geq 1 \tag{1.25}$$

$$\sum_{k=0}^{r}(-1)^k\binom{n}{k}\binom{n}{2r-k} = \frac{(-1)^r}{2}\left(\binom{n}{r}+\binom{n}{r}^2\right) \tag{1.26}$$

$$\sum_{k=0}^{r}(-1)^k\binom{2r}{k}\binom{2n-2r}{n-k} = \frac{(-1)^r}{2}\left(\binom{n}{r}+\binom{n}{r}^2\right)\frac{\binom{2n}{n}}{\binom{2n}{2r}} \tag{1.27}$$

$$\sum_{k=0}^{n}(-1)^k\binom{2n}{k}^2 = \frac{(-1)^n}{2}\left(\binom{2n}{n}+\binom{2n}{n}^2\right) \tag{1.28}$$

$$\sum_{k=0}^{[\frac{n}{2}]}\binom{2n}{2k}^2 = \frac{1}{4}\binom{4n}{2n}+\frac{(-1)^n}{4}\binom{2n}{n}+\frac{1+(-1)^n}{4}\binom{2n}{n}^2 \tag{1.29}$$

$$\sum_{k=0}^{[\frac{n-1}{2}]}\binom{2n}{2k+1}^2 = \frac{1}{4}\binom{4n}{2n}-\frac{(-1)^n}{4}\binom{2n}{n}+\frac{1-(-1)^n}{4}\binom{2n}{n}^2, \qquad n\geq 1 \tag{1.30}$$

$$\sum_{k=0}^{n}\binom{4n}{2k}^2 = \frac{1}{4}\binom{8n}{4n}+\frac{1}{4}\binom{4n}{2n}+\frac{1}{2}\binom{4n}{2n}^2 \tag{1.31}$$

$$\sum_{k=0}^{n}\binom{4n+2}{2k+1}^2 = \frac{1}{4}\binom{8n+4}{4n+2}+\frac{1}{4}\binom{4n+2}{2n+1}+\frac{1}{2}\binom{4n+2}{2n+1}^2 \tag{1.32}$$

1.4.3 Applications of Equation (1.19) to Expansions of $(1+x)^n(x-1)^n$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2n-k}{n}2^k = 0, \qquad \text{if } n \text{ is odd} \tag{1.33}$$

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{4n-k}{2n}2^k = (-1)^n\binom{2n}{n} \tag{1.34}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2 = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2n-k}{n}2^k = (-1)^{[\frac{n}{2}]}\binom{n}{[\frac{n}{2}]}\frac{1+(-1)^n}{2} \tag{1.35}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n+k}{n}\frac{1}{2^k}=0, \qquad \text{if } n \text{ is odd} \tag{1.36}$$

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{2n+k}{2n}\frac{1}{2^k}=(-1)^n\binom{2n}{n}\frac{1}{2^{2n}} \tag{1.37}$$

1.5 Expansions of $(1-x)^m(1+x)^{2n-m}$

$$\boldsymbol{\sum_{j=0}^{n}(-1)^j\binom{x}{j}\binom{2n-x}{n-j}=(-1)^n\sum_{j=0}^{n}(-1)^j\binom{2n-j}{n-j}\binom{2n-x}{j}2^j} \tag{1.38}$$

$$\boldsymbol{\sum_{j=0}^{n}(-1)^j\binom{x}{j}\binom{2n-x}{n-j}=\sum_{j=0}^{[\frac{n}{2}]}(-1)^j\binom{x}{j}\binom{2n-2x}{n-2j}} \tag{1.39}$$

$$\boldsymbol{\sum_{j=0}^{n}(-1)^j\binom{x}{j}\binom{2n-x}{n-j}=(-1)^n2^{2n}\binom{\frac{x-1}{2}}{n}} \tag{1.40}$$

$$\boldsymbol{\sum_{j=0}^{n}(-1)^j\binom{x}{j}\binom{2n-x}{n-j}=\frac{2^n}{n!}\prod_{k=0}^{n-1}(2k+1-x), \qquad n\geq 1} \tag{1.41}$$

$$\boldsymbol{\sum_{j=0}^{n}(-1)^j\binom{x}{j}\binom{2n-x}{n-j}=\sum_{k=0}^{n}\binom{-n-1}{n-k}\binom{2n-x}{k}2^k} \tag{1.42}$$

1.5.1 Applications of Equation (1.40)

$$\sum_{k=0}^{[\frac{n}{2}]}\binom{x}{2k}\binom{2n-x}{n-2k}=\frac{1}{2}\left(\binom{2n}{n}+(-1)^n2^{2n}\binom{\frac{x-1}{2}}{n}\right) \tag{1.43}$$

$$\sum_{k=0}^{[\frac{n-1}{2}]}\binom{x}{2k+1}\binom{2n-x}{n-2k-1}=\frac{1}{2}\left(\binom{2n}{n}-(-1)^n2^{2n}\binom{\frac{x-1}{2}}{n}\right), \qquad n\geq 1 \tag{1.44}$$

1.5.2 Special Cases of Equation (1.40)

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2 = (-1)^n 2^{2n}\binom{\frac{n-1}{2}}{n} = (-1)^{\left[\frac{n}{2}\right]}\binom{n}{\left[\frac{n}{2}\right]}\frac{1+(-1)^n}{2} \tag{1.45}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2n}{2k} = 2^{4n}\binom{\frac{n-1}{2}}{2n} \tag{1.46}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2n}{2k} = \frac{2^{4n}}{(2n)!}\prod_{k=0}^{2n-1}(2k+1-n), \qquad n \geq 1 \tag{1.47}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2n}{2k} = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{3n}{2n-k} = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{3n}{n+k} \tag{1.48}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n-\frac{1}{2}}{k}\binom{n+\frac{1}{2}}{n-k} = (-1)^n 2^{2n}\binom{\frac{2n-3}{4}}{n} \tag{1.49}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n-\frac{1}{2}}{k}\binom{n+\frac{1}{2}}{n-k} = \frac{\binom{2n}{n}^2(2n+1)}{2^{2n}}\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2\frac{1}{\binom{2k}{k}\binom{2n-2k}{n-k}(2k+1)} \tag{1.50}$$

Remark 1.7 *In the following identity, we evaluate $\left(\frac{1}{2}\right)!$ by the Gamma function, namely, $\left(\frac{1}{2}\right)! = \Gamma\left(\frac{3}{2}\right)$.*

$$\sum_{k=0}^{n}(-1)^k\binom{n-\frac{1}{2}}{k}\binom{n+\frac{1}{2}}{n-k} = \frac{\pi^2(2n+1)\binom{2n}{n}^2}{2^{4n+2}}\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2\binom{k}{\frac{1}{2}}\binom{n-k}{\frac{1}{2}}\frac{1}{2k+1} \tag{1.51}$$

$$\sum_{k=0}^{n}(-1)^k\binom{\frac{-1}{2}}{k}\binom{2n+\frac{1}{2}}{n-k} = \frac{2^n}{n!}\prod_{k=0}^{n-1}\left(2k+\frac{3}{2}\right), \qquad n \geq 1 \tag{1.52}$$

$$\sum_{k=0}^{n}\frac{\binom{2n}{n+k}\binom{2k}{k}}{\binom{2n+2k}{n+k}(2n+2k+1)} = \frac{2^{3n}}{(4n+1)\binom{4n}{2n}n!}\prod_{k=0}^{n-1}\left(2k+\frac{3}{2}\right), \qquad n \geq 1 \tag{1.53}$$

Remark 1.8 *The following two identities are found in C. van Ebbenhorst Tengbergen, "Über die Identitäten...etc." Nieuw. Arch. Wiskde., Vol. 18 (1943), pp. 1-7. We assume a is a nonnegative integer.*

$$\sum_{k=a}^{n}(-1)^k\binom{-2a}{k-a}\binom{2n}{n-k} = (-1)^a\frac{\binom{2n}{n}}{\binom{2a}{a}}\binom{n}{a}, \qquad n \geq a \geq 0 \tag{1.54}$$

$$\sum_{k=0}^{n}(-1)^k\binom{-2a}{k}\binom{2n+2a}{n-k} = \frac{\binom{2n+2a}{n+a}\binom{n+a}{a}}{\binom{2a}{a}}, \tag{1.55}$$

1.6 Expansions of $(1+x)^z(1-x)^{-z}$

Remark 1.9 *In the following section, we assume z is an arbitrary complex number.*

Mitlag-Leffler's Polynomials

$$\sum_{j=0}^{k}(-1)^j\binom{-z}{j}\binom{-z}{k-j} = \sum_{j=0}^{k-1}\binom{z}{j+1}\binom{k-1}{j}2^{j+1}, \qquad k \geq 1 \tag{1.56}$$

1.7 Expansions of $(1+z^p)^x(1+z)^{n-y-px}$

Remark 1.10 *In the following section, we assume z and y are arbitrary complex numbers, while p is a positive integer. A reference for these expansions is "Another Note on the Hermite Polynomials" by M. P. Drazin, American Math. Monthly, Vol. 64, 1957, pp. 89-91.*

$$\sum_{k=0}^{[\frac{n}{2}]}\binom{x}{k}\binom{y-2x}{n-2k} = \sum_{k=0}^{n}(-1)^k\binom{x}{k}\binom{y-2k}{n-k}2^k \tag{1.57}$$

$$\sum_{k=0}^{[\frac{n}{2}]}\binom{x}{k}\binom{x}{n-2k} = \sum_{k=0}^{n}(-1)^k\binom{x}{k}\binom{3x-2k}{n-k}2^k \tag{1.58}$$

$$\sum_{k=0}^{[\frac{n}{3}]}\binom{x}{k}\binom{y-3x}{n-3k} = \sum_{k=0}^{n}(-1)^k\binom{x}{k}\binom{y-2k}{n-k}3^k \tag{1.59}$$

$$(-1)^n \sum_{k=0}^{n} (-1)^k \binom{x}{k}\binom{3x-2k}{n-k} 3^k = \binom{x}{\frac{n}{3}} \frac{(-1)^{[\frac{n}{3}]} + (-1)^{[\frac{n+2}{3}]}}{2} \tag{1.60}$$
$$= \binom{x}{\frac{n}{3}} \frac{(-1)^{[\frac{n}{3}]} - (-1)^{[\frac{n-1}{3}]}}{2}$$

$$\sum_{k=0}^{[\frac{n}{3}]} \binom{x}{k}\binom{x}{n-3k} = \sum_{k=0}^{n} (-1)^k \binom{x}{k}\binom{4x-2k}{n-k} 3^k \tag{1.61}$$

1.8 Expansions of $(x^2+2x)^n$

Remark 1.11 *In the following section, we will assume k is a nonnegative integer, and y is an arbitrary complex number.*

$$\boldsymbol{\sum_{j=0}^{n} (-1)^j \binom{n}{j}\binom{2j}{k} = (-1)^n \binom{n}{k-n} 2^{2n-k}} \tag{1.62}$$

1.8.1 Applications of Equation (1.62)

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j}\binom{2j}{n} = (-1)^n 2^n \tag{1.63}$$

$$\sum_{j=0}^{n} \binom{n}{j}\binom{j}{k-j} 2^{2j} = 2^k \binom{2n}{k} \tag{1.64}$$

$$\sum_{j=0}^{[\frac{n}{2}]} \binom{n}{2j}\binom{2j}{j} \frac{1}{2^{2j}} = \frac{1}{2^n}\binom{2n}{n} \tag{1.65}$$

$$\sum_{j=0}^{n} \binom{n}{j}\binom{n-j}{k-2j} 2^{k-2j} = \binom{2n}{k} \tag{1.66}$$

1.8.2 Generalizations of Equation (1.64)

$$\sum_{k=0}^{n}\binom{x}{k}\binom{k}{n-k}2^{2k}=2^{n}\binom{2x}{n} \tag{1.67}$$

$$\sum_{k=0}^{[\frac{n}{2}]}\binom{x}{k}\binom{x-k}{n-2k}\frac{1}{2^{2k}}=2^{-n}\binom{2x}{n} \tag{1.68}$$

$$\sum_{k=0}^{[\frac{n}{2}]}\binom{x}{k}\binom{x-k}{n-2k}2^{n-2k}=\binom{2x}{n} \tag{1.69}$$

$$\sum_{k=0}^{n}\binom{x}{k}\binom{y+k}{n-k}2^{2k}=\sum_{k=0}^{n}\binom{2x}{k}\binom{y}{n-k}2^{k} \tag{1.70}$$

$$\sum_{k=0}^{n}\binom{x}{k}\binom{y}{n-k}2^{k}=\sum_{k=0}^{n}\binom{x}{k}\binom{x+y-k}{n-k} \tag{1.71}$$

$$\sum_{k=0}^{n}\binom{x}{k}\binom{y+k}{n-k}2^{2k}=\sum_{k=0}^{n}\binom{2x}{k}\binom{2x+y-k}{n-k} \tag{1.72}$$

Remark 1.12 *The following two identities are associated with Problem 3803 [1936, P.580; 1938, pp. 633-634] in The American Math. Monthly*

$$\sum_{k=0}^{n}(-1)^{k}\binom{x}{k}\binom{2x-2k}{n-k}2^{2k}=(-1)^{n}\binom{2x}{n} \tag{1.73}$$

$$\sum_{k=0}^{n}(-1)^{k}\binom{x}{k}\binom{4x-2k}{n-k}2^{2k}=(-1)^{\frac{n}{2}}\,\frac{1+(-1)^{n}}{2}\binom{2x}{\frac{n}{2}} \tag{1.74}$$

1.9 Expansions by I. J. Schwatt

Remark 1.13 *In the following section, we assume z is an arbitrary complex number. We also let $((z^n))$ denote the coefficient of z^n in the power series expansion. For a reference, see "Introduction to the Operations with Series" by I. J. Schwatt, Univ. of Pa. Press, 1924, pp. 93-94. A reprint of this book was issued in 1962 by Chelsea Publ. Co., New York, N. Y.*

$$\sum_{k=0}^{[\frac{n}{2}]}\binom{x}{k}\binom{x-k}{n-2k}2^{n-2k}=((z^n))(1+z)^{2x} \tag{1.75}$$

$$\sum_{k=0}^{[\frac{n}{2}]}\binom{x}{k}\binom{x-k}{n-2k}\frac{1}{2^{n-2k}}=\frac{1}{2^x}((z^n))(2z^2+z+2)^x \tag{1.76}$$

$$\sum_{k=0}^{[\frac{n}{2}]}\binom{x}{k}\binom{x-k}{n-2k}3^{n-2k}=((z^n))(z^2+3z+1)^x \tag{1.77}$$

1.10 Coefficient Comparisons Involving the Companion Binomial Theorem

Remark 1.14 *In this section, we will assume, unless otherwise specified, that a and r are arbitrary real or complex numbers.*

$$\boldsymbol{\sum_{k=0}^{n}\binom{a+k}{k}\binom{r+n-k}{n-k}=\binom{a+r+n+1}{n}} \tag{1.78}$$

$$\sum_{k=0}^{n}\binom{r+k}{k}\binom{r+n-k}{n-k}=\binom{2r+1+n}{n} \tag{1.79}$$

$$\sum_{k=0}^{[\frac{n-1}{2}]}\binom{r+k}{k}\binom{r+n-k}{n-k}=\frac{1}{2}\binom{2r+n+1}{n}-\frac{1+(-1)^n}{4}\binom{r+\left[\frac{n}{2}\right]}{\left[\frac{n}{2}\right]}^2, \qquad n\geq 1 \tag{1.80}$$

$$\sum_{k=0}^{n}\binom{r+k}{k}\binom{r+2n-k}{2n-k}=\frac{1}{2}\binom{2r+2n+1}{2n}+\frac{1}{2}\binom{r+n}{n}^2 \tag{1.81}$$

$$\sum_{k=0}^{n}\binom{r+k}{k}\binom{r+2n-k+1}{2n-k+1}=\frac{1}{2}\binom{2r+2n+2}{2n+1} \tag{1.82}$$

$$\boldsymbol{\sum_{i=0}^{k}(-1)^i\binom{n}{i}\binom{n+k-i}{n}=1} \tag{1.83}$$

2 Linear Algebra Techniques Applied to Series

Remark 2.1 *Throught this chapter, we assume, unless otherwise specified, that n is a nonnegative integer. We let x and z denote arbitrary real or complex numbers. If x is a real number, we let $[x]$ denote the integer part of x.*

2.1 Series Derived from M^n

Remark 2.2 *Throughout this section, we assume M is a 2×2 matrix of complex entries with two distinct eigenvalues, t_1 and t_2.*

$$\sum_{i=0}^{[\frac{n-1}{2}]} (-1)^i (t_1 t_2)^i (t_1 + t_2)^{n-2i-1} \binom{n-i-1}{i} = \frac{t_2^n - t_1^n}{t_2 - t_1} \tag{2.1}$$

2.1.1 Applications of Equation (2.1)

$$\sum_{i=0}^{[\frac{n}{2}]} (-1)^i \left(\frac{t}{(1+t)^2}\right)^i \binom{n-i}{i} = \frac{t^{n+1}-1}{(t-1)(1+t)^n}, \qquad t \neq 1 \tag{2.2}$$

$$\sum_{i=0}^{[\frac{n}{2}]} \binom{n-i}{i} z^i = \frac{(2z+1 \pm \sqrt{1+4z})^{n+1} + (-1)^n (2z)^{n+1}}{2^{\frac{n}{2}} (2z+1 \pm \sqrt{1+4z})^{\frac{n}{2}} (4z+1 \pm \sqrt{1+4z})} \tag{2.3}$$

$$\sum_{i=0}^{[\frac{n}{2}]} (-1)^i \binom{n-i}{i} \frac{1}{2^{2i}} = \frac{n+1}{2^n} \tag{2.4}$$

$$\sum_{k=0}^{n} (-1)^k \binom{n+k}{n-k} 2^{2k} = (-1)^n (2n+1) \tag{2.5}$$

$$\sum_{i=0}^{[\frac{n}{2}]} \binom{n-i}{i} 2^i = \frac{2^{n+1} + (-1)^n}{3} \tag{2.6}$$

$$\sum_{k=0}^{n} \binom{n+k}{n-k} \frac{1}{2^k} = \sum_{k=0}^{n} \binom{n+k}{2k} \frac{1}{2^k} = \frac{2^{2n+1}+1}{3 * 2^n} \tag{2.7}$$

$$\boldsymbol{\sum_{i=0}^{[\frac{n}{2}]}\binom{n-i}{i} = \frac{(1+\sqrt{5})^{n+1}-(1-\sqrt{5})^{n+1}}{2^{n+1}\sqrt{5}} = F_n,} \tag{2.8}$$

where F_n is the n^{th} Fibonacci number given by the recurrence $F_n = F_{n-1}+F_{n-2}, F_0 = 1, F_1 = 1.$

$$\sum_{k=0}^{[\frac{n}{2}]}(-1)^k\binom{n-k}{k} = \frac{(-1\underline{+}i\sqrt{3})^{n+1}-2^{n+1}}{2^{\frac{n}{2}}(-1\underline{+}i\sqrt{3})^{\frac{n}{2}}(-3\underline{+}i\sqrt{3})}, \qquad \text{where } i=\sqrt{-1} \tag{2.9}$$

$$\sum_{k=0}^{[\frac{n}{2}]}(-1)^k\binom{n-k}{k} = \frac{(-1)^{[\frac{n}{3}]}+(-1)^{[\frac{n+1}{3}]}}{2}, \qquad n=3m+a,\ (a=0,1,2,...) \tag{2.10}$$

Limiting Case of Equation (2.1)

$$\sum_{i=0}^{[\frac{n-1}{2}]}(-1)^i\binom{n-i-1}{i}\frac{1}{2^{2i}} = \frac{n}{2^{n-1}}, \qquad n\geq 1 \tag{2.11}$$

$$\boldsymbol{\sum_{i=0}^{n}(-1)^i(t_1t_2)^{n-i}(t_1+t_2)^{2i}\binom{n+i}{2i} = (-1)^n\frac{t_2^{2n+1}-t_1^{2n+1}}{t_2-t_1}} \tag{2.12}$$

$$\sum_{k=0}^{n}\binom{n+k}{2k}(t_1-t_2)^{2k}(t_1t_2)^{n-k} = \frac{t_1^{2n+1}+t_2^{2n+1}}{t_1+t_2} \tag{2.13}$$

$$\boldsymbol{\sum_{i=0}^{n}(-1)^i(t_1t_2)^{n-i}(t_1+t_2)^{2i+1}\binom{n+i+1}{2i+1} = (-1)^n\frac{t_2^{2n+2}-t_1^{2n+2}}{t_2-t_1}} \tag{2.14}$$

$$\sum_{i=0}^{n}(-1)^i(t_1t_2)^{n-i}(t_1+t_2)^{2i+1}\binom{n+i}{2i+1} = (-1)^{n-1}\frac{t_1t_2(t_2^{2n}-t_1^{2n})}{t_2-t_1} \tag{2.15}$$

$$\sum_{k=0}^{n}\binom{n+k}{2k+1}(t_1-t_2)^{2k+1}(t_1t_2)^{n-k-1} = \frac{t_1^{2n}-t_2^{2n}}{t_1+t_2} \tag{2.16}$$

$$\sum_{i=0}^{n}(-1)^i(t_1t_2)^{n-i}(t_1+t_2)^{2i+1}\left(\binom{n+i+1}{2i+1}+\binom{n+i}{2i+1}\right) = (-1)^n(t_2^{2n+1}+t_1^{2n+1}) \tag{2.17}$$

Remark 2.3 *A reference for Equation (2.18) is the Solution to Problem 4356 [1949, 479] in American Math. Monthly, April 1952, p. 268.*

$$\sum_{i=0}^{n}(-1)^i(t_1t_2)^{n-i}(t_1+t_2)^{2i+1}\binom{n+i}{2i}\frac{1}{2i+1}=(-1)^n\frac{t_2^{2n+1}+t_1^{2n+1}}{2n+1} \tag{2.18}$$

Limiting Case of Equation (2.18)

$$\sum_{i=0}^{n}(-1)^i\binom{n+i}{2i}\frac{2^{2i}}{2i+1}=\frac{(-1)^n}{2n+1} \tag{2.19}$$

$$\sum_{i=0}^{n}(-1)^i(t_1t_2)^{n-i}(t_1+t_2)^{2i-1}\binom{n+i-1}{2i-1}=(-1)^n\frac{t_2^{2n}-t_1^{2n}}{t_2-t_1} \tag{2.20}$$

2.2 Convolutions and Determinants

Remark 2.4 *For Equation (2.22), we given the following system of equations*

$$a_k=\sum_{j=0}^{k}b_jc_{k-j}, \tag{2.21}$$

where k is a nonnegative integer, and b_j is independent of k. We further assume that the a_k and b_j are known sequences (with $b_0\neq 0$). Our goal is to solve for c_n. This is done in Equation (2.22). Note the right side of Equation (2.22) involves the determinant of an $n\times n$ matrix.

$$\boldsymbol{c_n=\frac{1}{b_0^{n+1}}\begin{vmatrix} a_n & b_1 & b_2 & b_3 & \dots & b_n \\ a_{n-1} & b_0 & b_1 & b_2 & \dots & b_{n-1} \\ a_{n-2} & 0 & b_0 & b_1 & \dots & b_{n-2} \\ a_{n-3} & 0 & 0 & b_0 & \dots & b_{n-3} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_0 & 0 & 0 & 0 & \dots & b_0 \end{vmatrix}} \tag{2.22}$$

Remark 2.5 *For Equation (2.24), we are given the following system of equations*

$$a_k = \sum_{j=0}^{k} b_j^k c_{k-j}, \tag{2.23}$$

where k is a nonnegative integer, and b_j^k depends on k. Please note that the superscript does not denote a power of b, but only the dependence on k as well as j. We further assume that a_k and b_j^k are known (with $b_0^k \neq 0$). Our goal is to solve for c_n. This is done in Equation (2.24). Note the right side of Equation (2.24) involves the determinant of an $n \times n$ matrix.

$$\boldsymbol{c_n = \frac{1}{\prod_{i=0}^{n} b_0^i} \begin{vmatrix} a_n & b_1^n & b_2^n & b_3^n & \dots & b_n^n \\ a_{n-1} & b_0^{n-1} & b_1^{n-1} & b_2^{n-1} & \dots & b_{n-1}^{n-1} \\ a_{n-2} & 0 & b_0^{n-2} & b_1^{n-2} & \dots & b_{n-2}^{n-2} \\ a_{n-3} & 0 & 0 & b_0^{n-3} & \dots & b_{n-3}^{n-3} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_0 & 0 & 0 & 0 & \dots & b_0^0 \end{vmatrix}} \tag{2.24}$$

2.2.1 Application of Equation (2.24)

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} a_{n-j} = \begin{vmatrix} a_n & \binom{n}{1} & \binom{n}{2} & \binom{n}{3} & \dots & \binom{n}{n} \\ a_{n-1} & \binom{n-1}{0} & \binom{n-1}{1} & \binom{n-1}{2} & \dots & \binom{n-1}{n-1} \\ a_{n-2} & 0 & \binom{n-2}{0} & \binom{n-2}{1} & \dots & \binom{n-2}{n-2} \\ a_{n-3} & 0 & 0 & \binom{n-3}{0} & \dots & \binom{n-3}{n-3} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_0 & 0 & 0 & 0 & \dots & b_0 \end{vmatrix} \tag{2.25}$$

2.3 Application of Equation (2.24) to Reciprocal Expansions

Remark 2.6 *For Equation (2.27), we are given a power series representation for $f(x)$, namely, $f(x) = \sum_{i=0}^{\infty} a_i x^i$, with $a_0 \neq 0$. We assume α is an integer. Our goal is to calculate*

$$\frac{1}{(f(x))^\alpha} = \sum_{j=0}^{\infty} b_j^\alpha x^j, \tag{2.26}$$

where the superscripts on the $b's$ are not exponents, but just an indicator of the entry in an array of numbers (See Remark 2.5). These calculations are done in Equation (2.27). Note the right side of Equation (2.27) is the determinant of an $n \times n$ matrix.

$$\boldsymbol{b_0^\alpha = \frac{1}{a_0^\alpha},} \qquad \text{where the } \boldsymbol{\alpha} \text{ subscript on } \boldsymbol{a_0} \text{ is an exponent,} \tag{2.27}$$

and if $n \geq 1$,

$$\boldsymbol{b_n^\alpha = \frac{1}{a_0^{n+\alpha} n!} \begin{vmatrix} 0 & (\alpha+n-1)a_1 & (2\alpha+n-2)a_2 & .. & .. & \alpha n a_n \\ 0 & (n-1)a_0 & (\alpha+n-2)a_1 & .. & .. & \alpha(n-1)a_{n-1} \\ 0 & 0 & (n-2)a_0 & .. & .. & \alpha(n-2)a_{n-2} \\ 0 & 0 & 0 & (n-3)a_0 & ... & \alpha(n-3)a_{n-3} \\ ... & ... & ... & ... & ... & ... \\ 1 & 0 & 0 & .. & .. & 1 \end{vmatrix}}$$

2.3.1 An Alternative Form of Equation (2.27)

$$b_n^\alpha = \frac{(-1)^n}{a_0^{n+\alpha} n!} D,$$

where D is the determinant of the following $(n-1) \times (n-1)$ matrix

$$D = \begin{vmatrix} \alpha a_1 & a_0 & 0 & 0 & ... & 0 \\ 2\alpha a_2 & (\alpha+1)a_1 & 2a_0 & 0 & ... & 0 \\ 3\alpha a_3 & (2\alpha+1)a_2 & (\alpha+2)a_1 & 3a_0 & ... & 0 \\ ... & ... & ... & ... & ... & ... \\ n\alpha a_n & ((n-1)\alpha+1)a_{n-1} & ((n-2)\alpha+2)a_{n-2} & ((n-3)\alpha+3)a_{n-3} & ... & (\alpha+n-1)a_1 \end{vmatrix}$$

2.4 Generalized Cauchy Convolution

Remark 2.7 *For Equation (2.29), we are given a power series* $\sum_{i=0}^{\infty} a_i x^i$. *Our goal is to calculate product of this power series, namely*

$$\left(\sum_{i=0}^{\infty} a_i x^i\right)^{\alpha} = \sum_{n=0}^{\infty} B_n^{\alpha} x^n, \tag{2.28}$$

where α *is an integer,* $a_0 \neq 0$ *if* α *is a negative integer, and the superscript on the* B *is not an exponent (see Remark 2.5). Equation (2.29) provides a solution for* B_n^{α}. *Note* $B_0^{\alpha} = a_0^{\alpha}$, *while if* $n \geq 1$,

$$\boldsymbol{B_n^{\alpha} = \sum_{k=1}^{n} \binom{\alpha}{k} a_0^{\alpha-k} f_k(n),} \tag{2.29}$$

where the superscript on a_0 *denotes a power, and*

$$f_k(n) = \sum_{\substack{\sum_{i=1}^{k} j_i = n \\ 1 \leq j_i \leq n}} a_{j_1} a_{j_2} a_{j_3} \ldots a_{j_k}. \tag{2.30}$$

2.5 Inverse Function Expansions

Remark 2.8 *Suppose we are given a power series* $y = \sum_{i=1}^{\infty} a_i x^i$, *with* $a_1 \neq 0$. *Our goal is to invert this series, namely to find* $x = \sum_{\alpha=1}^{\infty} b_{\alpha} y^{\alpha}$, *where* $b_1 = \frac{1}{f_1(1)} = \frac{1}{a_1}$, *and for* $n \geq 1$,

$$\boldsymbol{b_n = \frac{(-1)^{n-1}}{a_1^{\frac{n(n+1)}{2}}} \begin{vmatrix} f_1(2) & f_2(2) & 0 & 0 & \ldots & 0 \\ f_1(3) & f_2(3) & f_3(3) & 0 & \ldots & 0 \\ f_1(4) & f_2(4) & f_3(4) & f_4(4) & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ f_1(n) & f_2(n) & f_3(n) & f_4(n) & \ldots & f_{n-1}(n) \end{vmatrix},} \tag{2.31}$$

with $f_k(n)$ *defined by Equation (2.30).*

3 Annihilation Coefficients

Remark 3.1 *In this chapter, we work with the series*

$$\sum_{k=1}^{n} c_k (x-k)^{n-k+1}, \tag{3.1}$$

where x is any real or complex number for which Equation (3.1) is defined. Our goal is to determine the c_k such that all terms in the sum, except x^n and the constant term, vanish simultaneously. All the formulas in this chapter are derived by this annihilation procedure. Also, we assume n is a positive integer, and m is a nonnegative integer.

$$\boldsymbol{\sum_{r=0}^{n} \binom{n}{r} (x-r)^{n-r} (r+1)^{r-1} = (x+1)^n} \tag{3.2}$$

$$\sum_{r=0}^{n} (-1)^r \binom{n}{r} (x+r)^{n-r} (r+1)^{r-1} = (x-1)^n \tag{3.3}$$

$$\sum_{r=0}^{n} \binom{n}{r} (r+1)^{r-1} (-r)^{n-r-1} = 1 - \frac{1}{n} \tag{3.4}$$

$$\sum_{r=0}^{m} \binom{n}{r} \binom{n-r}{m} (r+1)^{r-1} (x-r)^{n-r-m} = \binom{n}{m} (x+1)^{n-m} \tag{3.5}$$

$$\sum_{k=1}^{n} \binom{n}{k} k^k (n-k+1)^{n-k-1} = n(n+1)^{n-1} \tag{3.6}$$

4 Kummer Series Transformation

Remark 4.1 *In this chapter, we start with the series* $\sum_{k=a}^{n} f(k)$*, where both* n *and* a *are nonnegative integers. For an arbitrary function* $\varphi(n)$*, we have*

$$\sum_{k=a}^{n} f(k) = \sum_{k=a}^{n} \left(1 - \frac{f(n)\varphi(k)}{\varphi(n)f(k)}\right) f(k) + \frac{f(n)}{\varphi(n)} \sum_{k=a}^{n} \varphi(k) \tag{4.1}$$

Suppose that

i. $\lim_{n\to\infty} \frac{f(n)}{\varphi(n)} = K \neq 0$

ii. $S = \sum_{k=a}^{\infty} f(k)$ *coverges*

iii. $C = \sum_{k=a}^{\infty} \varphi(k)$ *converges.*

Then, Equation (4.1) implies

$$S = \sum_{k=a}^{\infty} \left(1 - K\frac{\varphi(k)}{f(k)}\right) f(k) + KC. \tag{4.2}$$

Equation (4.2) is known as the Kummer Series Transformation. A reference for this material is Konrad Knopp's "Theorie und Anwendung der unendlichen Reihen" Fourth Edition, Berlin, 1947. See Chapter VIII, P. 269.

4.1 Applications of the Kummer Series Transformation

$$\frac{\pi^2}{6} = \sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + 6\sum_{k=1}^{\infty} \frac{1}{k^2(k+1)(k+2)(k+3)} \tag{4.3}$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{k=1}^{r} \frac{1}{k^2} + r!\sum_{k=1}^{\infty} \frac{(k-1)!}{k(k+r)!}, \qquad r \geq 1 \tag{4.4}$$

$$\sum_{k=1}^{\infty} \frac{1}{\binom{k+r}{k} k^2} = \sum_{k=1}^{\infty} \frac{1}{k^2} - \sum_{k=1}^{r} \frac{1}{k^2}, \qquad r \geq 1 \tag{4.5}$$

Remark 4.2 *In the following identity, we assume* r *is a positive integer.*

$$\sum_{k=1}^{\infty} \frac{1}{k^r} = \sum_{k=1}^{\infty} \frac{(r+k)! - k^r k!}{(r+k)!k^r} + \frac{1}{(r-1)r!} \tag{4.6}$$

4.1.1 Knopp Applications of the Kummer Series Transformation

Remark 4.3 *In section, we assume p is a positive integer, and α is arbitrary real or complex number, excluding the set of nonpositive integers. Define*

$$S_p(\alpha) = \sum_{k=0}^{\infty} \left(\frac{1}{(k+\alpha)(k+\alpha+1)...(k+\alpha+p-1)} \right)^2. \tag{4.7}$$

Then,

$$S_p(\alpha) = \frac{(\alpha + \frac{3}{2}p - 1)}{(2p-1)(\alpha(\alpha+1)...(\alpha+p-1))^2} + \frac{p^3}{2(2p-1)} S_{p+1}(\alpha) \tag{4.8}$$

$$S_{n+1}(1) = \frac{2(2n-1)}{n^3} S_n - \frac{3}{n^2 n!^2} \tag{4.9}$$

$$S_{n+1}(1) = \frac{\pi^2}{6n!^2} \binom{2n}{n} - \frac{3}{n!^2} \binom{2n}{n} \sum_{j=1}^{n} \frac{1}{\binom{2j}{j} j^2} \tag{4.10}$$

$$\sum_{k=1}^{\infty} \frac{1}{\binom{k+n}{k}^2 k^2} = \frac{\pi^2}{6} \binom{2n}{n} - 3\binom{2n}{n} \sum_{j=1}^{n} \frac{1}{\binom{2j}{j} j^2} \tag{4.11}$$

Limiting Case of Equation (4.11)

$$\sum_{k=1}^{\infty} \frac{1}{\binom{2k}{k} k^2} = \frac{\pi^2}{18} \tag{4.12}$$

5 Series From Logarithmic Differentiation

Remark 5.1 *Throughout this chapter, we will assume x is an arbitrary real or complex number, while n is a nonnegative integer.*

5.1 Basic Logarithmic Differentiation Formula

Assume a is a nonnegative integer. If

$$F_n(x) = \prod_{k=a}^{n} (1 + u_k(x)), \tag{5.1}$$

then

$$\frac{F_n'(x)}{F_n(x)} = \sum_{k=a}^{n} \frac{u_k'(x)}{1 + u_k(x)}, \tag{5.2}$$

where $F_n'(x)$ is the derivative of $F_n(x)$ with respect to x.

5.2 Applications of the Basic Formula

$$\sum_{k=0}^{n} \frac{2^k x^{2^k}}{1 + x^{2^k}} = \frac{x}{1-x} - \frac{2^{n+1} x^{2^{n+1}}}{1 - x^{2^{n+1}}}, \qquad x \neq 1 \tag{5.3}$$

$$\sum_{k=0}^{\infty} \frac{2^k x^{2^k}}{1 + x^{2^k}} = \frac{x}{1-x}, \qquad |x| < 1 \tag{5.4}$$

$$\sum_{k=0}^{n-1} \frac{2^k}{1 + x^{2^k}} = \frac{1}{x-1} + \frac{2^n}{1 - x^{2^n}}, \qquad x \neq 1, \qquad n \geq 1 \tag{5.5}$$

$$\sum_{k=0}^{\infty} \frac{2^k}{1 + x^{2^k}} = \frac{1}{x-1}, \qquad |x| > 1 \tag{5.6}$$

$$\sum_{k=0}^{n-1} \frac{x^{2k} x^{2^k}}{(1 + x^{2^k})^2} = \frac{x}{(x-1)^2} - \frac{2^{2n} x^{2^n}}{(1 - x^{2^n})^2}, \qquad x \neq 1, \qquad n \geq 1 \tag{5.7}$$

Remark 5.2 *The following identity is Problem 728 of Math. Magazine, Vol. 42, May 1969, P. 153*

$$\sum_{k=0}^{\infty} \frac{2^k x^{2^k} - 2^{k+1}}{x^{2^{k+1}} - x^{2^k} + 1} = \frac{x+2}{x^2+x+1}, \qquad |x| > 1 \tag{5.8}$$

Remark 5.3 *In the following two identities, we assume r is a positive integer.*

$$\sum_{k=0}^{n-1} r^k \frac{\sum_{i=1}^{r-1} i x^{i r^k}}{\sum_{i=0}^{r-1} x^{i r^k}} = \frac{x}{1-x} - \frac{r^n x^{r^n}}{1-x^{r^n}}, \qquad x \neq 1, \qquad r \geq 2 \tag{5.9}$$

$$\sum_{k=0}^{\infty} 3^k \frac{x^{3^k} + 2x^{2*3^k}}{1 + x^{3^k} + x^{2*3^k}} = \frac{x}{1-x}, \qquad |x| < 1 \tag{5.10}$$

$$\sum_{k=1}^{n} \frac{1}{2^k} \tan\left(\frac{x}{2^k}\right) = \frac{1}{2^n} \cot\left(\frac{x}{2^n}\right) - \cot x \tag{5.11}$$

6 Vandermonde Convolution

6.1 Integral Version of the Vandermonde Convolution

Assume r and q are nonnegative integers. Then,

$$\boldsymbol{\sum_{i=0}^{k} \binom{r}{i} \binom{q}{k-i} = \binom{r+q}{k}.} \tag{6.1}$$

Equation (6.1) is known as the (integral) Vandermonde Convolution.

6.1.1 Applications of Equation (6.1)

Remark 6.1 *Throughout this section, we will assume r, q, and n are nonnegative integers. Also recall that for real x, $[x]$ is the integer part of x.*

$$\sum_{i=0}^{n} \binom{n}{i} \binom{r+q-n}{r-i} = \binom{r+q}{r} \tag{6.2}$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n} \tag{6.3}$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 k = \frac{n}{2} \sum_{k=0}^{n} \binom{n}{k}^2 = \frac{n}{2} \binom{2n}{n} \tag{6.4}$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 k^2 = n^2 \binom{2n-2}{n-1} \tag{6.5}$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 k^3 = \frac{n^2(n+1)}{2} \binom{2n-2}{n-1} \tag{6.6}$$

Remark 6.2 *The following two identities are from Problem 3414, proposed by B. C. Wong, in The American Math. Monthly, March 1930, Vol. 37, No. 3.*

$$\sum_{k=0}^{n} \binom{n}{k}^2 (n-2k)^2 = 2n \binom{2n-2}{n-1} \tag{6.7}$$

$$\sum_{k=0}^{[\frac{n-1}{2}]} \binom{n}{k}^2 (n-2k)^2 = n \binom{2n-2}{n-1}, \qquad n \geq 1 \tag{6.8}$$

6.2 Generalized Version of the Vandermonde Convolution

Assume x and y are arbitrary real or complex numbers. Assume n is a nonegative integer. Then,

$$\sum_{k=0}^{n} \binom{x}{k} \binom{y}{n-k} = \binom{x+y}{n}. \tag{6.9}$$

Equation (6.9) is the (generalized) Vandermonde Convolution.

6.2.1 Specific Evaluations of Equation (6.9) Using the $\frac{-1}{2}$ Transformation

$$\sum_{k=0}^{n} \binom{2n-2k}{n-k} \binom{2k}{k} = 2^{2n} \tag{6.10}$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 \frac{1}{\binom{2n}{2k}} = \frac{2^{2n}}{\binom{2n}{n}} \tag{6.11}$$

$$\sum_{k=0}^{n} \binom{2n-2k}{n-k}\binom{2k}{k}\frac{1}{2k-1} = \begin{cases} 0, & n \geq 1 \\ -1, & n = 0 \end{cases} \tag{6.12}$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 \frac{1}{\binom{2n}{2k}(2k-1)} = \begin{cases} 0, & n \geq 1 \\ -1, & n = 0 \end{cases} \tag{6.13}$$

$$\sum_{k=0}^{n} \binom{2k}{k}\binom{2n-2k}{n-k}\frac{1}{(2k-1)(2n-2k-1)} = \begin{cases} 1, & n = 0 \\ -4, & n = 1 \\ 0, & n \geq 2 \end{cases} \tag{6.14}$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{2k}{k}\frac{1}{2^{2k}} = \frac{1}{2^{2n}}\binom{2n}{n} \tag{6.15}$$

$$\sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k}\binom{2k}{k}\frac{2k+1}{2^{2k}} = \frac{1}{2^{2n}(2n-1)}\binom{2n}{n} \tag{6.16}$$

6.2.2 Various Applications of Equation (6.9)

Remark 6.3 *For the rest of Chapter 6, we assume, unless otherwise specified, that α and r are nonnegative integers. We also assume x, y, and z are arbitrary real or complex numbers.*

$$\boldsymbol{\sum_{k=0}^{n} \binom{x}{k}\binom{y}{n-k}\binom{k}{\alpha} = \binom{x}{\alpha}\binom{y+x-\alpha}{n-\alpha}} \tag{6.17}$$

Remark 6.4 *In the following two identities, we assume $f(x)$ is a polynomial of degree n, namely $f(x) = \sum_{i=0}^{n} a_i x^i = \sum_{j=0}^{n} b_j \binom{x}{j}$.*

$$\sum_{k=0}^{n} \binom{x}{k}\binom{y}{n-k} f(k) = \sum_{j=0}^{n} b_j \binom{x}{j}\binom{x+y-j}{n-j} \tag{6.18}$$

$$\sum_{k=0}^{n}\binom{x}{k}\binom{y}{n-k}f(k+z)=\sum_{j=0}^{n}b_j\sum_{\alpha=0}^{j}\binom{z}{j-\alpha}\binom{x}{\alpha}\binom{y+x-\alpha}{n-\alpha} \tag{6.19}$$

$$\sum_{k=0}^{n}\binom{n}{\alpha}\binom{x+\alpha}{n}=\sum_{j=0}^{n}\binom{n}{j}\binom{x}{j}2^j \tag{6.20}$$

Remark 6.5 *In the following identity, due to Laplace, assume u is a real or complex number, $u \neq -1$.*

$$\boldsymbol{\sum_{k=0}^{n}\binom{x-n+k}{k}\frac{u^k}{(1+u)^k}=\frac{1}{(1+u)^n}\sum_{j=0}^{n}\binom{x+1}{j}u^j} \tag{6.21}$$

$$\sum_{k=0}^{n}\binom{n+k}{k}((1-x)^{n+1}x^k+x^{n+1}(1-x)^k)=1, \qquad x\neq 0,1 \tag{6.22}$$

$$\sum_{k=0}^{n}\binom{n+k}{k}\frac{(x-1)^{n+1}-(x-1)^k}{x^k}=x^{n+1}, \qquad x\neq 0,1 \tag{6.23}$$

$$\boldsymbol{\sum_{k=0}^{n}(-1)^k\binom{x}{k}\binom{2n-k}{n}=\binom{2n-x}{n}} \tag{6.24}$$

$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-k}{n}2^{-2k}=\binom{2n+\frac{1}{2}}{n}=\frac{1}{2^{2n}}\binom{4n+1}{2n} \tag{6.25}$$

$$\boldsymbol{\sum_{k=0}^{n}\binom{n}{k}\binom{x}{k}\frac{2^{2k}}{(2k+1)\binom{2k}{k}}=\frac{2^{2n}}{(2n+1)\binom{2n}{n}}\binom{x+n+\frac{1}{2}}{n}} \tag{6.26}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{x+k}{k}\frac{2^{2k}}{(2k+1)\binom{2k}{k}}=\frac{2^{2n}}{(2n+1)\binom{2n}{n}}\binom{n-x-\frac{1}{2}}{n} \tag{6.27}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n+k}{k}\frac{2^{2k}}{(2k+1)\binom{2k}{k}}=\frac{(-1)^n}{2n+1} \tag{6.28}$$

$$\sum_{k=0}^{n}\binom{x}{k}\binom{x}{2n-k}=\sum_{k=0}^{n}\binom{x}{n-k}\binom{x}{n+k}=\frac{1}{2}\left(\binom{2x}{2n}+\binom{x}{n}^2\right) \tag{6.29}$$

$$\begin{aligned}\sum_{k=0}^{n-1}\binom{x}{k}\binom{x}{2n-1-k}&=\sum_{k=0}^{n-1}\binom{x}{n-1-k}\binom{x}{n+k}\\&=\frac{1}{2}\binom{2x}{2n-1},\qquad n\geq 1\end{aligned} \tag{6.30}$$

Remark 6.6 *The following three identities are found in H. F. Sandham's Problem 4519 [1953, 47] of The American Math. Monthly, April 1954, Vol. 4, pp. 265-266.*

$$\sum_{k=0}^{n-1}\frac{\binom{n-1}{k}\binom{x-1}{k}}{\binom{-n-1}{k}\binom{-x-1}{k}}=\frac{(-1)^n}{2}\frac{\binom{-x-\frac{1}{2}}{n-1}}{\binom{-x-1}{n-1}\binom{\frac{-1}{2}}{n}},\qquad n\geq 1 \tag{6.31}$$

$$\sum_{k=0}^{n-1}(-1)^k\frac{\binom{-x-1-k}{n-1-k}\binom{x-1}{k}}{\binom{n+k}{k}}=\frac{(-1)^n}{2}\frac{\binom{-x-\frac{1}{2}}{n-1}}{\binom{\frac{-1}{2}}{n}},\qquad n\geq 1 \tag{6.32}$$

$$\sum_{k=0}^{n-1}\binom{x+n-1}{n-1-k}\binom{x+n-1}{n+k}=\frac{-\binom{-x-\frac{1}{2}}{n-1}\binom{x-1+n}{n}}{2\binom{\frac{-1}{2}}{n}},\qquad n\geq 1 \tag{6.33}$$

$$\sum_{k=0}^{r-1}\binom{z}{k}y^{r-k-1}=\sum_{j=1}^{r}\binom{z-j}{r-j}(y+1)^{j-1},\qquad r\geq 1 \tag{6.34}$$

$$\sum_{k=0}^{r-1}\binom{z}{k}\frac{x^{r-k}}{r-k}=\sum_{j=1}^{r}\binom{z-j}{r-j}\frac{(x+1)^j-1}{j},\qquad r\geq 1 \tag{6.35}$$

$$\sum_{k=0}^{n}\binom{n}{k}\frac{x^{r+k}}{r+k}=\sum_{j=1}^{r}(-1)^{r-j}\binom{r-1}{r-j}\frac{(x+1)^{j+n}-1}{j+n},\qquad r\geq 1 \tag{6.36}$$

$$\sum_{k=0}^{n}\binom{x}{k}f(k)=\sum_{j=0}^{n}(-1)^j\binom{n-x}{j}\sum_{k=0}^{n-j}\binom{n-j}{k}f(k+j) \tag{6.37}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n-x}{k}y^k(1+y)^{n-k}=\sum_{k=0}^{n}\binom{x}{k}y^k \tag{6.38}$$

$$\sum_{k=0}^{n}\binom{x}{k}\binom{y}{k}=\sum_{k=0}^{n}(-1)^k\binom{n-x}{k}\binom{y+n-k}{n} \tag{6.39}$$

Remark 6.7 *The following indentity, due to H. L. Krall, is from Part 2 of Problem 783, Math. Magazine, Vol. 44, 1971, p.41.*

$$\boldsymbol{\sum_{k=0}^{n}\frac{1}{\binom{n}{k}}\binom{x}{k}\binom{y}{k}\binom{z}{n-k}\binom{x+y+z-k}{n-k}=\binom{x+z}{n}\binom{y+z}{n}} \tag{6.40}$$

$$\sum_{k=0}^{n}\binom{y}{k}\binom{x+y+z-k}{n-k}\binom{x+z-n}{x-k}=\binom{x+z}{x}\binom{y+z}{n} \tag{6.41}$$

$$\sum_{k=0}^{n}\frac{\binom{x}{k}\binom{y}{k}\binom{z}{n-k}}{\binom{x+y+z}{k}}=\frac{\binom{x+z}{n}\binom{y+z}{n}}{\binom{x+y+z}{n}} \tag{6.42}$$

$$\sum_{k=0}^{n}\frac{\binom{x}{k}\binom{y}{k}\binom{x+y+z-k}{n-k}}{\binom{z-n+k}{k}}=\frac{\binom{x+z}{n}\binom{y+z}{n}}{\binom{z}{n}} \tag{6.43}$$

6.3 Equation (6.9) in Reciprocal Binomial Identities

$$\boldsymbol{\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{\binom{z}{j}}{\binom{y}{j}}=\frac{\binom{y-z}{n}}{\binom{y}{n}}} \tag{6.44}$$

6.3.1 Specific Evaluations of Equation (6.44)

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}\binom{z+j}{j}\frac{1}{\binom{x+j}{j}}=\frac{\binom{-x-1-z}{n}}{\binom{-x-1}{n}} \tag{6.45}$$

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{j+1}{\binom{x+j}{j}}=\frac{\binom{-x+1}{n}}{\binom{-x-1}{n}} \tag{6.46}$$

$$\sum_{j=0}^{n}\binom{n}{j}\binom{z}{j}\frac{1}{\binom{x+j}{j}}=\frac{\binom{x+z+n}{n}}{\binom{x+n}{n}} \tag{6.47}$$

$$\sum_{j=0}^{n}\binom{n}{j}\binom{z}{j}\frac{1}{j+1}=\frac{1}{n+1}\binom{z+n+1}{n} \tag{6.48}$$

$$\sum_{j=0}^{2n}\binom{2n}{j}\binom{2j}{j}\frac{1}{2^{2j}\binom{n+j}{j}}=\frac{1}{2^{4n}}\frac{\binom{6n}{3n}}{\binom{2n}{n}} \tag{6.49}$$

Remark 6.8 *The following identity is due to Harry Bateman. The reader is referred to The Harry Bateman Manuscripit Project by Erdelyi, Magnus, and Oberhettinger. In particular, see Higher Transcendental Functions, Vol. I, [Section 2.5.3, P. 86], McGraw-Hill, 1953.*

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{\binom{z}{j}}{\binom{-z+n-1}{j}}=\frac{\binom{2z}{n}}{\binom{z}{n}} \tag{6.50}$$

6.3.2 Applications of Equation (6.44)

Remark 6.9 *In the following two identities, we will assume x and z are not values which make the denominator equal to zero.*

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\frac{1}{\binom{x+k}{k}}\sum_{j=1}^{k}\frac{1}{j+x}=\frac{n}{(x+n)^2}, \qquad n\geq 1 \tag{6.51}$$

$$\sum_{k=1}^{n}\binom{n}{k}\binom{z}{k}\frac{1}{\binom{x+k}{k}}\sum_{j=1}^{k}\frac{1}{j+x}=\frac{\binom{x+z+n}{n}}{\binom{x+n}{n}}\left(\sum_{j=1}^{n}\frac{1}{j+x}-\sum_{j=1}^{n}\frac{1}{j+x+z}\right), \qquad n\geq 1 \tag{6.52}$$

$$\sum_{k=1}^{n}\binom{n}{k}^2\sum_{j=1}^{k}\frac{1}{j}=\binom{2n}{n}\left(\sum_{j=1}^{n}\frac{1}{j}-\sum_{j=n+1}^{2n}\frac{1}{j}\right), \qquad n\geq 1 \tag{6.53}$$

$$\sum_{k=1}^{n}(-1)^k\binom{n}{k}\binom{n+k-1}{k}\sum_{j=1}^{k}\frac{1}{j}=\frac{(-1)^n}{n}, \qquad n\geq 1 \tag{6.54}$$

Remark 6.10 *The following identity, due to R. R. Goldberg, is found in Problem 4805 of The American Math. Monthly, Vol. 65, 1958, P. 633.*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n+k-1}{k}\sum_{j=1}^{n+k-1}\frac{1}{j}=\frac{(-1)^n}{n}, \qquad n\geq 1 \tag{6.55}$$

6.4 Application of Equation (6.9) to $\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{x}{k}\binom{x}{2n-k}$

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{x}{k}\binom{x}{2n-k}=\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{x+k}{k}\binom{x+2n-k}{2n-k}$$
$$=(-1)^n\binom{2n}{n}\binom{x+n}{2n}=\binom{x}{n}\binom{-x-1}{n} \tag{6.56}$$

6.4.1 Specific Evaluations of Equation (6.56)

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{n+k}{n}\binom{3n-k}{n}=(-1)^n\binom{2n}{n} \tag{6.57}$$

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{3n+k}{k}\binom{5n-k}{2n-k}=(-1)^n\binom{2n}{n}\binom{4n}{2n} \tag{6.58}$$

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{3n}{k}\binom{3n}{2n-k}=(-1)^n\binom{2n}{n}\binom{4n}{2n} \tag{6.59}$$

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{2k}{k}\binom{4n-2k}{2n-k}=\binom{2n}{n}^2 \tag{6.60}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2k}{k}\binom{2n-2k}{n-k}=\frac{1+(-1)^n}{2}\binom{n}{\left[\frac{n}{2}\right]}^2 \tag{6.61}$$

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{2n+\frac{1}{2}}{k}\binom{2n+\frac{1}{2}}{2n-k}=(-1)^n\binom{2n}{n}\binom{3n+\frac{1}{2}}{2n} \tag{6.62}$$

$$\sum_{k=0}^{2n}(-1)^k\binom{4n+1}{2k}\binom{4n+1}{4n-2k}\binom{2k}{k}\binom{4n-2k}{2n-k}\frac{1}{\binom{2n}{k}}$$
$$=(-1)^n\frac{n+1}{2n+1}\binom{6n+1}{2n}\binom{4n+1}{3n} \tag{6.63}$$

6.4.2 Applications of Equation (6.56)

Remark 6.11 *The following identity is found in L. Carlitz's "Note on a formula of Szily", Scripta Mathematica, Vol. 18 (1952), pp. 249-253.*

$$\sum_{k=0}^{r}(-1)^k\binom{2\alpha}{k}\binom{2r-2\alpha}{r-k}^2=(-1)^\alpha\frac{\binom{2\alpha}{\alpha}\binom{2r-\alpha}{r}^2}{\binom{2r-\alpha}{\alpha}} \tag{6.64}$$

$$\boldsymbol{\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{x}{k}\binom{x}{n-k}=\frac{(-1)^n+1}{2}(-1)^{\left[\frac{n}{2}\right]}\binom{n}{\left[\frac{n}{2}\right]}\binom{x+\left[\frac{n}{2}\right]}{n}} \tag{6.65}$$

$$\boldsymbol{\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{x}{k}\binom{x}{n-k}=\frac{(-1)^n+1}{2}\binom{x}{\frac{n}{2}}\binom{-x-1}{\frac{n}{2}}} \tag{6.66}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{x+k}{k}\binom{x+n-k}{n-k}=\frac{(-1)^n+1}{2}(-1)^{\left[\frac{n}{2}\right]}\binom{n}{\left[\frac{n}{2}\right]}\binom{x+\left[\frac{n}{2}\right]}{n} \tag{6.67}$$

Remark 6.12 *In the following identity, we assume x is a real or complex which is not a negative integer. We evaulate $x!=\Gamma(x+1)$.*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2\binom{2x-n}{x-k}=\frac{(-1)^n+1}{2}(-1)^{\left[\frac{n}{2}\right]}\binom{n}{\left[\frac{n}{2}\right]}\binom{x+\left[\frac{n}{2}\right]}{n}\frac{\binom{2x}{x}}{\binom{2x}{n}} \tag{6.68}$$

6.5 The Index Shift Formula and Equation (6.9)

$$\boldsymbol{\sum_{k=0}^{n}\binom{n}{k}\binom{x}{k+\alpha}=\binom{n+x}{n+\alpha}} \tag{6.69}$$

$$\sum_{k=0}^{n}\binom{n}{k}\binom{x}{k-\alpha}=\binom{n+x}{n-\alpha} \tag{6.70}$$

6.5.1 Applications of Equation (6.69)

Remark 6.13 *For the following two identities, a reference is P. Tardy's, "Sopra alcune formole relativa ai coefficienti binomiali", Battaglini's Journal, Vol. 3 (1865), pp. 1-3. In the second identity, we assume* a *is a nonnegative integer.*

$$\sum_{k=0}^{n}\binom{x+1}{k}\binom{x+1}{k+1}\binom{x-k}{n-k}=\binom{x+1}{n+1}\binom{x+n+2}{n} \tag{6.71}$$

$$\sum_{k=0}^{n}\binom{x+1}{k}\binom{x+a-1}{k+a-1}\binom{x-k}{n-k}=\binom{x+a-1}{n+a-1}\binom{x+n+a}{n} \tag{6.72}$$

$$\sum_{j=0}^{n}\binom{n}{j}\binom{x}{j}z^j=\sum_{\alpha=0}^{n}\binom{n}{\alpha}\binom{x+n-\alpha}{n}(z-1)^\alpha \tag{6.73}$$

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}\binom{x}{j}=\sum_{\alpha=0}^{n}(-1)^\alpha\binom{n}{\alpha}\binom{x+n-\alpha}{n}2^\alpha \tag{6.74}$$

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}\binom{2j}{j}\left(\frac{z}{4}\right)^j=\frac{1}{2^{2n}}\sum_{\alpha=0}^{n}(-1)^\alpha\binom{2n-2\alpha}{n-\alpha}\binom{2\alpha}{\alpha}(z-1)^\alpha \tag{6.75}$$

$$2^{2n}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\binom{2j}{j}=\sum_{\alpha=0}^{n}(-1)^\alpha\binom{2n-2\alpha}{n-\alpha}\binom{2\alpha}{\alpha}3^\alpha \tag{6.76}$$

Remark 6.14 *The following identity is from the Mathematical Reviews, Vol. 17, No 5., May 1956, pp. 459-460.*

$$\sum_{k=0}^{n}\binom{n}{k}^2\binom{x+k}{2n}=\binom{x}{n}^2 \tag{6.77}$$

Remark 6.15 *The following identity is from the Mathematical Reviews, Vol. 18, No 1., January 1957, P. 4.*

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{x+2n-k}{2n} = \binom{x+n}{n}^2 \tag{6.78}$$

$$\sum_{k=0}^{n} \binom{x}{k}^2 \binom{n+2x-k}{n-k} = \binom{x+n}{n}^2 \tag{6.79}$$

$$\sum_{k=0}^{n} \binom{x-n}{k}^2 \binom{2x-n-k}{n-k} = \binom{x}{n}^2 \tag{6.80}$$

Remark 6.16 *For the following identity see "Remark on a Note of P. Turán" by T. S. Nanjundiah in American Math Monthly, Vol. 65, No. 5, May 1958, P. 354.*

$$\sum_{k=0}^{n} \binom{m-x+y}{k}\binom{n+x-y}{n-k}\binom{x+k}{m+n} = \binom{x}{m}\binom{y}{n}, \qquad m \textit{ integeral, } m \geq 0 \tag{6.81}$$

Remark 6.17 *The following identity is a special case of Saalschütz's Theorem. See the review by L. Carlitz in Math Reviews, Vol. 18, No. 1. January 1957, P. 4.*

$$\sum_{k=0}^{n} \binom{n}{k}\binom{\alpha}{k}\binom{x+n+\alpha-k}{n+\alpha} = \binom{x+\alpha}{\alpha}\binom{x+n}{n} \tag{6.82}$$

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j}\binom{-x+r-1}{r-j}\binom{x-r-n}{\alpha-r-j} = (-1)^r \binom{\alpha+n}{\alpha-r}\binom{x}{\alpha} \tag{6.83}$$

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j}\binom{-x-1}{j}\binom{x-n}{n-j} = \binom{2n}{n}\binom{x}{n} \tag{6.84}$$

$$\sum_{j=0}^{n} \binom{n}{j}^2 \binom{2n+j}{2n} = \sum_{j=0}^{n} \binom{n}{j}^2 \binom{3n-j}{2n} = \binom{2n}{n}^2 \tag{6.85}$$

$$\sum_{j=0}^{n} \binom{n}{j} \binom{3n+j}{j} \binom{2n}{n-j} = \binom{2n}{n} \binom{3n}{n} \tag{6.86}$$

$$\sum_{j=0}^{n} \binom{n}{j}^2 \binom{3n+j}{2n} = \binom{3n}{n}^2 \tag{6.87}$$

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j}^2 \binom{2j}{j} \binom{4n-2j}{2n-j} \frac{1}{\binom{2n}{j}} = \binom{2n}{n}^2 \tag{6.88}$$

$$\sum_{j=0}^{n} \binom{n}{j}^2 \binom{4n+2j+1}{2j} \frac{1}{(2j+1)\binom{2n+j}{j}} = \frac{1}{2n+1} \binom{4n+1}{2n} \tag{6.89}$$

$$\sum_{j=0}^{2n} (-1)^j \binom{2n}{j} \binom{-x-1}{j} \binom{x-2n}{\alpha-j} = \binom{\alpha+2n}{\alpha} \binom{x}{\alpha} \tag{6.90}$$

$$\sum_{j=0}^{n} \binom{2n}{j} \binom{3n+j}{3n} \binom{n}{j} = \binom{3n}{n}^2 \tag{6.91}$$

$$\sum_{k=0}^{n} \binom{n}{k} \frac{\binom{x}{k}\binom{y}{k+r}}{\binom{x+y+n}{k}} = \frac{\binom{x+r+n}{n}\binom{y+n}{n+r}}{\binom{x+y+n}{n}} \tag{6.92}$$

Remark 6.18 *The following identity solves a problem on Page 122, Vol. 29, 1947, of Norsk Matematisk Tidsskrift.*

$$\sum_{k=0}^{n} \binom{n}{k} \frac{\binom{x}{k}\binom{y}{k}}{\binom{x+y+n}{k}} = \frac{\binom{x+n}{n}\binom{y+n}{n}}{\binom{x+y+n}{n}} \tag{6.93}$$

$$\sum_{k=0}^{n} \binom{n}{k} \binom{x}{k} \binom{x+n+k+\alpha}{k+\alpha} = \binom{x+\alpha+n}{n} \binom{x+\alpha+n}{n+\alpha} \tag{6.94}$$

$$\sum_{k=0}^{n} \binom{n}{k} \binom{r}{k} \binom{x+n+r+k}{n+r} = \binom{x+n+r}{n} \binom{x+n+r}{r} \tag{6.95}$$

6.6 The Minus One Tranformation Applied to Equation (6.9)

$$\sum_{k=0}^{n}\binom{x+k}{k}\binom{y+n-k}{n-k}=\binom{x+y+n+1}{n} \tag{6.96}$$

$$\sum_{k=\alpha}^{n-\beta}\binom{k}{\alpha}\binom{n-k}{\beta}=\binom{n+1}{\alpha+\beta+1}, \qquad n-\beta\geq\alpha \tag{6.97}$$

7 Alternating Convolutions

Remark 7.1 *Throughout this chapter, we assume, unless otherwise specified, that n and r are nonnegative integers, while x, y, and z are arbitrary real or complex numbers.*

7.1 Evaluation of $\sum_{k=0}^{2n}(-1)^k\binom{x}{k}\binom{x}{2n-k}$

$$\sum_{k=0}^{2n}(-1)^k\binom{x}{k}\binom{x}{2n-k}=(-1)^n\binom{x}{n} \tag{7.1}$$

Remark 7.2 *In the following identity, we evaluate $x! = \Gamma(x+1)$.*

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{2x}{k+x}\binom{k+x}{2n}=(-1)^n\binom{x}{n}\binom{2x}{x} \tag{7.2}$$

7.1.1 Specific Evaluations and Applications of Equation (7.1)

$$\sum_{k=0}^{2n}(-1)^k\binom{2k}{k}\binom{4n-2k}{2n-k}=2^{2n}\binom{2n}{n} \tag{7.3}$$

$$\sum_{k=0}^{n}(-1)^k\binom{2k}{k}\binom{2n-2k}{n-k}=\frac{1+(-1)^n}{2}\binom{n}{\frac{n}{2}}2^n \tag{7.4}$$

$$\sum_{k=0}^{n}\binom{4k}{2k}\binom{4n-4k}{2n-2k}=2^{4n-1}+2^{2n-1}\binom{2n}{n} \tag{7.5}$$

$$\sum_{k=0}^{n-1} \binom{4k+2}{2k+1}\binom{4n-2k-2}{2n-2k-1} = 2^{4n-1} - 2^{2n-1}\binom{2n}{n}, \qquad n \geq 1 \tag{7.6}$$

$$\sum_{k=0}^{n} (-1)^k \binom{x}{n-k}\binom{x}{n+k} = \frac{1}{2}\left(\binom{x}{n} + \binom{x}{n}^2\right) \tag{7.7}$$

$$\sum_{k=0}^{n} (-1)^k \frac{\binom{n}{k}\binom{x}{k}}{\binom{n+k}{k}\binom{x+k}{k}} = \frac{1}{2}\left(1 + \frac{1}{\binom{x+n}{n}}\right) \tag{7.8}$$

7.2 Evaluation of $\sum_{k=0}^{n}(-1)^k\binom{x}{k}\binom{x}{n-k}$

Remark 7.3 *Recall that for real x, $[x]$ denotes the integer part of x.*

$$\boldsymbol{\sum_{k=0}^{n} (-1)^k \binom{x}{k}\binom{x}{n-k} = \frac{(-1)^n + 1}{2}(-1)^{[\frac{n}{2}]}\binom{x}{[\frac{n}{2}]}} \tag{7.9}$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{2x-n}{x-k} = \frac{(-1)^n + 1}{2}(-1)^{[\frac{n}{2}]}\binom{x}{[\frac{n}{2}]}\frac{\binom{2x}{x}}{\binom{2x}{n}}, \qquad \textit{see Remark 7.2} \tag{7.10}$$

$$\sum_{k=0}^{n} (-1)^k \binom{x+k}{k}\binom{x+n-k}{n-k} = (-1)^n \binom{x+\frac{n}{2}}{\frac{n}{2}}\frac{1+(-1)^n}{2} \tag{7.11}$$

7.2.1 Applications of Equation (7.9)

$$\sum_{k=0}^{n} (-1)^k \binom{y}{n-k}\binom{x+y}{k} = \sum_{\alpha=0}^{n} (-1)^{\alpha+[\frac{n-\alpha}{2}]}\binom{x}{\alpha}\frac{(-1)^{n-\alpha}+1}{2}\binom{y}{[\frac{n-\alpha}{2}]} \tag{7.12}$$

$$\sum_{k=r}^{n-r} (-1)^k \binom{k}{r}\binom{n-k}{r} = (-1)^r \frac{1+(-1)^n}{2}\binom{\frac{n}{2}}{r}, \qquad n \geq 2r \tag{7.13}$$

$$\sum_{k=0}^{n} (-1)^k \binom{x}{k}\binom{y}{n-k} = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \binom{x}{k}\binom{y-x}{n-2k} \tag{7.14}$$

$$\sum_{k=0}^{[\frac{n}{2}]} (-1)^k \binom{x}{k}\binom{-x}{n-2k} = (-1)^n \binom{x}{n} \tag{7.15}$$

Remark 7.4 *In the following two identities, we evaluate $z!$ by $\Gamma(z+1)$, whenever z is a real(complex) number which is not a negative integer. A reference for these two identities is H. Bateman's, "Higher Transcendental Functions", Vol I, Chapter II [editor W. Magnus].*

$$\sum_{k=0}^{n} \binom{n}{k}\binom{z}{k}\frac{1}{\binom{-z+n-1}{k}} = \frac{2^n (z-n)!\sqrt{\pi}}{\left(z-\frac{n}{2}\right)!\left(\frac{-n}{2}-\frac{1}{2}\right)!} \tag{7.16}$$

$$\sum_{k=0}^{n}(-1)^k \binom{z}{k}\binom{z}{n-k} = \binom{z}{n}\frac{2^n (z-n)!\sqrt{\pi}}{\left(z-\frac{n}{2}\right)!\left(\frac{-n}{2}-\frac{1}{2}\right)!} \tag{7.17}$$

$$\sum_{k=0}^{[\frac{n}{2}]} \binom{x}{2k}\binom{x}{n-2k} = \frac{1}{2}\binom{2x}{n} + \frac{(-1)^{[\frac{n}{2}]}}{2}\binom{x}{[\frac{n}{2}]}\frac{(-1)^n+1}{2} \tag{7.18}$$

$$\sum_{k=0}^{[\frac{n-1}{2}]} \binom{x}{2k+1}\binom{x}{n-2k-1} = \frac{1}{2}\binom{2x}{n} - \frac{(-1)^{[\frac{n}{2}]}}{2}\binom{x}{[\frac{n}{2}]}\frac{(-1)^n+1}{2}, \qquad n \geq 1 \tag{7.19}$$

$$\sum_{k=0}^{[\frac{n}{2}]} \binom{n}{2k}^2 = \frac{1}{2}\binom{2n}{n} + \frac{(-1)^{[\frac{n}{2}]}}{2}\binom{n}{[\frac{n}{2}]}\frac{(-1)^n+1}{2} \tag{7.20}$$

$$\sum_{k=0}^{[\frac{n-1}{2}]} \binom{n}{2k+1}^2 = \frac{1}{2}\binom{2n}{n} - \frac{(-1)^{[\frac{n}{2}]}}{2}\binom{n}{[\frac{n}{2}]}\frac{(-1)^n+1}{2}, \qquad n \geq 1 \tag{7.21}$$

7.3 Quotient Identities Involving Equations (6.9) and (7.9)

Remark 7.5 *Throughout this section, the reader may let x be a real or complex number. Then $x!$ is evaluated as $\Gamma(x+1)$.*

7.3.1 First Example of a Reciprocal Identity

$$\boldsymbol{\sum_{k=0}^{n} \frac{\binom{n}{k}}{\binom{n+2x}{k+x}} f(k) = \frac{(-1)^n}{\binom{2x}{x}\binom{n+2x}{n}} \sum_{k=0}^{n} \binom{-x-1}{k}\binom{-x-1}{n-k} f(k)} \tag{7.22}$$

Specific Evaluations of Equation (7.22)

$$\sum_{k=0}^{n} \frac{\binom{n}{k}}{\binom{n+2x}{k+x}} = \frac{2x+n+1}{(2x+1)\binom{2x}{x}} \tag{7.23}$$

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{1}{\binom{n+2x}{k+x}} = \frac{(-1)^{n+[\frac{n}{2}]}}{\binom{2x}{x}\binom{2x+n}{n}} \frac{1+(-1)^n}{2} \binom{-x-1}{\frac{n}{2}} \tag{7.24}$$

$$\sum_{k=0}^{2n}(-1)^k \binom{2n}{k} \frac{1}{\binom{2x+2n}{x+k}} = \frac{\binom{x+n}{n}}{\binom{2x}{x}\binom{2x+2n}{2n}} = \frac{\binom{2n}{n}}{\binom{x+n}{n}\binom{2x+2n}{x+n}} \tag{7.25}$$

7.3.2 Second Example of a Reciprocal Identity

$$\sum_{k=0}^{2n}(-1)^k \binom{2n}{k} \binom{2n+2x}{k+x} = (-1)^n \binom{2n}{n} \frac{\binom{2n+2x}{n+x}}{\binom{2n+x}{n}} = (-1)^n \binom{2n}{n} \frac{\binom{2n+2x}{n}}{\binom{n+x}{n}} \tag{7.26}$$

$$\sum_{k=0}^{2n} \binom{2n}{k} \binom{2n+2x}{k+x} = \binom{4n+2x}{2n+x} \tag{7.27}$$

7.3.3 Third Example of a Reciprocal Identity

$$\boldsymbol{\sum_{k=0}^{n} \binom{n}{k} \binom{n+2x}{k+x} f(k) = \frac{\binom{2n+2x}{n+x}}{\binom{2n+2x}{n}} \sum_{k=0}^{n} \binom{n+x}{k} \binom{n+x}{n-k} f(k)} \tag{7.28}$$

Specific Evaluations of Equation (7.28)

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k} \binom{n+2x}{k+x} = \frac{\binom{2n+2x}{n+x}}{\binom{2n+2x}{n}} (-1)^{[\frac{n}{2}]} \frac{1+(-1)^n}{2} \binom{-n-x}{\frac{n}{2}} \tag{7.29}$$

$$\sum_{k=0}^{n} \binom{n}{k} \binom{n+2x}{k+x} = \binom{2n+2x}{n+x} \tag{7.30}$$

7.4 Evaluation of $\sum_{k=0}^{2n}(-1)^k\binom{y}{2n-k}\binom{z}{k}$

$$\sum_{k=0}^{2n}(-1)^k\binom{y}{2n-k}\binom{z}{k}=\sum_{\alpha=0}^{n}(-1)^\alpha\binom{y}{\alpha}\binom{z-y}{2n-2\alpha} \tag{7.31}$$

$$\sum_{k=0}^{2n}\binom{\frac{-1}{2}}{k}=\frac{1}{2^{4n}}\sum_{\alpha}^{n}\binom{2\alpha}{\alpha}\binom{4n-4\alpha}{2n-2\alpha}2^{2\alpha} \tag{7.32}$$

8 An Introduction to the n^{th} Difference Operator

Remark 8.1 *Throughout this chapter, we will assume, unless otherwise specified, that n and r are nonnegative integers, x is a real or complex number, and h is nonzero real or complex number.*

8.1 Definition and Basic Properties of $\Delta_h^n f(x)$

8.1.1 Definition of $\Delta_h^n f(x)$

$$\boldsymbol{\Delta_h^n f(x)=\sum_{k=0}^{n}(-1)^{n+k}\binom{n}{k}\frac{f(x+kh)}{h^n}} \tag{8.1}$$

Relationship to Derivative

$$\frac{d^n f(x)}{dx^n}=\lim_{h\to 0}\sum_{k=0}^{n}(-1)^{n+k}\binom{n}{h}\frac{f(x+kh)}{h^n} \tag{8.2}$$

Linearity Properties of $\boldsymbol{\Delta_h^n}$

$$\Delta_h^1(f(x)+\varphi(x))=\Delta_h^1 f(x)+\Delta_h^1\varphi(x) \tag{8.3}$$

$$\Delta_h^n\Delta_h^r f(x)=\Delta_h^r\Delta_h^n f(x)=\Delta_h^{n+r}f(x) \tag{8.4}$$

$$\Delta_h^1(cf(x))=c\Delta_h^1 f(x),\qquad c\text{ is a constant} \tag{8.5}$$

Two Basic Examples

Remark 8.2 *In the following two examples, we assume a and α are nonzero real or complex numbers.*

$$\Delta_h^n a^x=a^x\left(\frac{a^h-1}{h}\right)^n \tag{8.6}$$

$$\Delta_h^n(1+\alpha h)^{\frac{x}{h}}=(1+\alpha h)^{\frac{x}{h}}\alpha^n \tag{8.7}$$

8.1.2 Inversion of Equation (8.1)

$$\boldsymbol{f(x+nh) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \Delta_h^k f(x)} \tag{8.8}$$

8.1.3 Calculation of $\Delta_1^k \binom{x}{k}^p \frac{k!^p}{x^{kp}}$

Remark 8.3 *Throughout this section, we assume p is an integer while x is a nonzero real or complex number.*

$$\sum_{k=0}^{n} \binom{x}{k}^p \frac{k!^p}{x^{(k+1)p}} ((x-k)^p - x^p) = \binom{x}{n+1}^p \frac{(n+1)!^p}{x^{(n+1)p}} - 1 \tag{8.9}$$

$$\sum_{k=0}^{n} \binom{x}{k} \frac{kk!}{x^{k+1}} = 1 - \binom{x}{n+1} \frac{(n+1)!}{x^{n+1}} \tag{8.10}$$

Remark 8.4 *The following two identities are found in D. Steinberg's "Combinatorial Derivations of Two Identities", Mathematics Magazine, Vol. 31, No.4, 1958, pp. 207-9.*

$$\sum_{k=1}^{m} \frac{m!km^{n-k-1}}{(m-k)!} = m^n, \qquad \textit{for } m < n+1 \tag{8.11}$$

$$\sum_{k=1}^{n-1} \frac{m!km^{n-k-1}}{(m-k)!} + \frac{m!}{(m-n)!} = m^n, \qquad \textit{for } m > n \tag{8.12}$$

$$\sum_{k=0}^{n} kk! = (n+1)! - 1 \tag{8.13}$$

$$\sum_{k=0}^{n} \binom{2k}{k} \frac{kk!}{2^k} = \binom{2n+2}{n+1} \frac{(n+1)!}{2^{n+2}} - \frac{1}{2} \tag{8.14}$$

$$\sum_{k=0}^{n} (-1)^k \binom{-x+k-1}{k} \frac{kk!}{x^k} = x + (-1)^n \binom{-x+n}{n+1} \frac{(n+1)!}{x^n} \tag{8.15}$$

$$\sum_{k=0}^{n}\binom{n+k}{k}\frac{kk!}{(n+1)^k}=\binom{2n+1}{n+1}\frac{(n+1)!}{(n+1)^n}-n-1 \tag{8.16}$$

$$\sum_{k=0}^{n}\frac{x^{k+1}k}{x(x-k)k!\binom{x}{k}}=\frac{x^{n+1}}{(n+1)!\binom{x}{n+1}}-1 \tag{8.17}$$

$$\sum_{k=0}^{n}\frac{k}{(k+1)!}=1-\frac{1}{(n+1)!} \tag{8.18}$$

8.1.4 Euler's Tranformation

If

$$\boldsymbol{(-1)^n\Delta_1^n f(0)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}f(k),} \tag{8.19}$$

then

$$\boldsymbol{\sum_{n=0}^{\infty}(-1)^n f(n)=\sum_{n=0}^{\infty}(-1)^n\frac{\Delta_1^n f(0)}{2^{n+1}}.} \tag{8.20}$$

8.1.5 Alternative Definition of the n^{th} Difference Operator

Definition of $\boldsymbol{\tilde{\Delta}_h^n}$

$$\boldsymbol{\tilde{\Delta}_h^n=\sum_{k=0}^{n}(-1)^k\binom{n}{k}f(x+kh)} \tag{8.21}$$

Inversion of Equation (8.21)

$$\boldsymbol{f(x+nh)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\tilde{\Delta}_h^n f(x)} \tag{8.22}$$

$$\sum_{j=0}^{n}\tilde{\Delta}_h^n f(x)=\sum_{k=0}^{n}(-1)^k\binom{n+1}{k+1}f(x+kh) \tag{8.23}$$

$$\sum_{j=0}^{n}f(x+jh)=\sum_{k=0}^{n}(-1)^k\binom{n+1}{k+1}\tilde{\Delta}_h^k f(x) \tag{8.24}$$

Examples of Equations (8.21) and (8.22)

$$\sum_{j=1}^{n} \frac{1}{j} = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{1}{k}, \qquad n \geq 1 \tag{8.25}$$

$$\frac{1}{n} = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \sum_{j=1}^{k} \frac{1}{j}, \qquad n \geq 1 \tag{8.26}$$

Example of Equation (8.23) Let

$$f(n) = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} F(k). \tag{8.27}$$

Then,

$$\sum_{i=1}^{n} f(i) = \sum_{k=1}^{n} (-1)^{k-1} \binom{n+1}{k+1} F(k). \tag{8.28}$$

Generalization of Equations (8.21) and (8.22): The Inversion Pair Theorem

$$\boldsymbol{\sum_{k=\alpha}^{n} (-1)^{k} \binom{k}{\alpha} f(k) = (-1)^{n} g(\alpha)} \tag{8.29}$$

if and only if

$$\boldsymbol{\sum_{k=\alpha}^{n} (-1)^{k} \binom{k}{\alpha} g(k) = (-1)^{n} f(\alpha)} \tag{8.30}$$

8.2 The Shift Operator E_h^n

8.2.1 Definition of E_h^n

$$\boldsymbol{E_h^n f(x) = f(x + nh)} \tag{8.31}$$

8.2.2 Relationships Between the Shift and Difference Operators

$$\boldsymbol{\Delta_h^n f(x) = \frac{(-1)^n}{h^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} E_h^k f(x)} \tag{8.32}$$

$$\Delta_h^n f(x) = \frac{1}{h^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} E_h^{n-k} f(x) \tag{8.33}$$

$$\boldsymbol{E_h^n f(x) = \sum_{k=0}^{n} \binom{n}{k} h^k \Delta_h^k f(x)} \tag{8.34}$$

$$E_h^1 \Delta_h^1 f(x) = \Delta_h^1 E_h^1 f(x) = \frac{f(x + 2h) - f(x + h)}{h} \tag{8.35}$$

8.2.3 Product Formula for $\Delta_h^n uv$

$$\Delta_h^n uv = \sum_{k=0}^{n} \binom{n}{k} \Delta_h^k u \Delta_h^{n-k} (E_h^k v) \tag{8.36}$$

Application of Equation (8.36)

Remark 8.5 *In the following example, we assume a is a nonzero real or complex number.*

$$\Delta_h^n (e^{ax} f(x)) = e^{ax} \sum_{k=0}^{n} \binom{n}{k} e^{ahk} \Delta_h^k f(x) \left(\frac{e^{ah} - 1}{h} \right)^{n-k} \tag{8.37}$$

8.2.4 Recurrence Formula for $\Delta_h^n f(x + h)$

$$\Delta_h^n f(x + h) = h \Delta_h^{n+1} f(x) + \Delta_h^n f(x) \tag{8.38}$$

9 Calculations Involving $(-1)^n\Delta_1^n\frac{1}{x}|_{x=r}$

Remark 9.1 *Throughout this chapter, we assume n and m are nonnegative integers. We also assume, unless otherwise specified r, z, x, and a are real or complex numbers for which the Gamma function is defined.*

$$\boldsymbol{\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{k+r}=\frac{n!(r-1)!}{(n+r)!}=\beta(r,n+1)} \tag{9.1}$$

$$\boldsymbol{\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{ak+1}=\frac{1}{\binom{n+\frac{1}{a}}{n}}} \tag{9.2}$$

9.1 Specfic Evaluations of Equation (9.2)

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{2k+1}=\frac{2^{2n}}{(2n+1)\binom{2n}{n}} \tag{9.3}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{2n-2k+1}=(-1)^n\frac{2^{2n}n!^2}{(2n+1)!} \tag{9.4}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{2^{2k}k!^2}{(2k+1)!}=\frac{1}{2n+1} \tag{9.5}$$

9.2 Applications of the Inversion Pair to Equation (9.2)

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{\binom{k+\frac{1}{a}}{k}}=\frac{1}{an+1}, \qquad a\neq 0 \tag{9.6}$$

$$\boldsymbol{\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{\binom{k+m}{k}}=\frac{m}{n+m}} \tag{9.7}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{\binom{k+\frac{1}{3}}{k}}=\frac{1}{3n+1} \tag{9.8}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n+r}{n-k}=\binom{n+r-1}{n} \tag{9.9}$$

$$\sum_{j=0}^{n}\frac{1}{aj+1}=\sum_{k=0}^{n}(-1)^k\binom{n+1}{k+1}\frac{1}{\binom{k+\frac{1}{a}}{k}},\qquad a\neq 0 \tag{9.10}$$

$$\sum_{j=1}^{n}\frac{1}{j+a}=\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\frac{1}{k\binom{k+a}{a}} \tag{9.11}$$

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\sum_{j=1}^{k}\frac{1}{j+a}=\frac{1}{n\binom{n+a}{a}} \tag{9.12}$$

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\sum_{j=1}^{k}\frac{1}{j+a}=\frac{1}{a},\qquad a\neq 0 \tag{9.13}$$

9.2.1 Applications of Equation (9.7)

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\sum_{j=1}^{m}\frac{1}{k+j}=\frac{1}{n}\left(1-\frac{1}{\binom{n+m}{n}}\right),\qquad n\geq 1, m\geq 1 \tag{9.14}$$

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{1}{z+k}=\frac{e^{-1}}{z}\sum_{j=0}^{\infty}\frac{1}{\binom{z+j}{j}j!},\qquad z\neq 0 \tag{9.15}$$

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{1}{n+1+k}=n!\left(1-\frac{1}{e}\sum_{k=0}^{n}\frac{1}{k!}\right) \tag{9.16}$$

$$\sum_{k=1}^{\infty}\frac{1}{\binom{n+k}{k}k!}=n!\left(e-\sum_{k=0}^{n}\frac{1}{k!}\right) \tag{9.17}$$

$$\sum_{k=0}^{\infty}\frac{x^{k+1}}{\binom{z+k}{k}}=\sum_{j=0}^{\infty}(-1)^j\frac{z}{z+j}\left(\frac{x}{1-x}\right)^{j+1},\qquad |x|<1,\qquad |\frac{x}{1-x}|<1 \tag{9.18}$$

Two Restatements of Equation (9.18)

$$\sum_{j=0}^{\infty} \frac{x^{j+1}}{z+j} = -\sum_{k=0}^{\infty} \left(\frac{x}{x-1}\right)^{k+1} \frac{1}{z\binom{z+k}{k}}, \qquad |\frac{x}{1-x}| < 1 \tag{9.19}$$

$$\sum_{k=0}^{\infty} \frac{x^{k+1}}{\binom{z}{k}} = \sum_{j=0}^{\infty} \left(\frac{x}{x+1}\right)^{j+1} \frac{z+1}{z+1-j}, \qquad |\frac{x}{1+x}| < 1 \tag{9.20}$$

Three Specific Evaluations of Equation (9.18)

$$\sum_{k=0}^{\infty} (-1)^k \frac{1}{\binom{z+k}{k}} = \sum_{j=0}^{\infty} \frac{z}{2^{j+1}(z+j)} \tag{9.21}$$

$$\sum_{k=0}^{\infty} \frac{1}{\binom{x+k}{k} 2^k} = 2\sum_{j=0}^{\infty} (-1)^j \frac{x}{x+j} \tag{9.22}$$

$$\sum_{j=0}^{\infty} \frac{(-1)^j}{x+j} = \frac{1}{x} \sum_{k=0}^{\infty} \frac{1}{\binom{x+k}{k} 2^{k+1}}, \qquad x \neq 0 \tag{9.23}$$

10 Euler's Finite Difference Theorem

Remark 10.1 *Throughout this chapter, we assume n, j, m, p, α, and r are nonnegative integers. We also assume x, z, a, y, and b are real or complex numbers.*

10.1 Statement of Euler's Finite Difference Theorem

$$\boldsymbol{\sum_{k=0}^{n} (-1)^k \binom{n}{k} k^j = \begin{cases} 0, & 0 \leq j < n \\ (-1)^n n!, & j = n \end{cases}} \tag{10.1}$$

10.2 Variations of Equation (10.1)

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} k^{n+1} = (-1)^n (n+1)! \frac{n}{2} \tag{10.2}$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} k^{n+2} = (-1)^n (n+2)! \frac{n(3n+1)}{24} \tag{10.3}$$

10.3 Polynomial Extension of Equation (10.1)

Remark 10.2 *Given a polynomial $\sum_{j=0}^{r} a_j k^j$ of degree r in k, with a_j free of k, Equation (10.1) implies*

$$\sum_{j=0}^{r} a_j \sum_{k=0}^{n} (-1)^k \binom{n}{k} k^j = \begin{cases} 0, & r < n \\ (-1)^n n! a_n, & r = n \end{cases} \tag{10.4}$$

10.3.1 Applications of Equation (10.4)

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (x-k)^r = \begin{cases} 0, & r < n \\ n!, & r = n \end{cases} \tag{10.5}$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (a-bk)^r = \begin{cases} 0, & r < n \\ b^n n!, & r = n \end{cases} \tag{10.6}$$

Remark 10.3 *Througout the remainder of this chapter, we will assume, unless otherwise specified, that $f(x)$ is a polynomial of degree n in x, namely $f(x) = \sum_{i=0}^{n} a_i x^i$.*

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} f(x-kz) = \begin{cases} 0, & n < m \\ a_m z^m m!, & n = m \end{cases} \tag{10.7}$$

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} \binom{mj}{r} = \begin{cases} 0, & r < n \\ (-1)^n m^n, & r = n, \qquad m \text{ not necessarily an integer} \end{cases} \tag{10.8}$$

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} \binom{j}{r} = \begin{cases} 0, & r < n \\ (-1)^n, & r = n \end{cases} \tag{10.9}$$

Two Applications of Equation (10.9)

$$\sum_{k=1}^{n} k^p = \sum_{k=1}^{p} (-1)^k \binom{n+1}{k+1} \sum_{j=1}^{k} (-1)^k \binom{k}{j} j^p, \qquad p \geq 1, n \geq 1 \tag{10.10}$$

$$\sum_{k=1}^{n} k^p = \sum_{k=0}^{p} (-1)^k \binom{n}{k+1} \sum_{j=0}^{k} (-1)^k \binom{k}{j} (j+1)^p, \qquad n \geq 1 \tag{10.11}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{r+k}{r}=\begin{cases}0, & r<n\\(-1)^n, & r=n\end{cases}\tag{10.12}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{y+xk}{r}=\begin{cases}0, & r<n\\(-1)^n x^n, & r=n,\end{cases}\tag{10.13}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{\binom{2xk}{2\alpha}}{\binom{xk}{\alpha}}=\begin{cases}0, & 0\le\alpha<n\\(-1)^n x^n\frac{2^{2n}}{\binom{2n}{n}}, & \alpha=n\end{cases}\tag{10.14}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{x+k}{r}=(-1)^n\binom{x}{r-n}\tag{10.15}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{\alpha+k-1}{\alpha}=\begin{cases}0, & \alpha<n\\(-1)^n\binom{\alpha-1}{n-1}, & \alpha\ge n\end{cases}\tag{10.16}$$

$$\Delta_1^n\binom{x}{r}=\binom{x}{r-n}\tag{10.17}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{x+k}{r+k}=(-1)^n\binom{x}{r+n}\tag{10.18}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{-x+\alpha-1}{k+r}\binom{x}{\alpha-r-k}=(-1)^r\binom{\alpha+n}{\alpha-r}\binom{x}{\alpha}\tag{10.19}$$

Three Applications of Equation (10.15)

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{x-k}{j}=\binom{x-n}{j-n}\tag{10.20}$$

Remark 10.4 *The following two identities are due to the work of H. Bateman and T. Ono. For references see Question 2221 in Nouv. Ann. de Math, Vol.4. No. 14, 1915, pp. 191-192, and Problem 2238, Nouv. Ann. de Math, Vol.4. No. 15, 1915, pp. 56-57. Also see "Notes on Binomial Coefficients" by H. Bateman, P. 54.*

$$\sum_{k=0}^{n-a}\binom{n-a}{k}\frac{\binom{a-\frac{1}{2}+k}{a+k}}{\binom{n-\frac{1}{2}}{a+k}}=\frac{\binom{a-\frac{1}{2}}{a}}{\binom{n-\frac{1}{2}}{n}},\qquad a \text{ a nonnegative integer, } a\le n \tag{10.21}$$

$$\sum_{k=a}^{n}\binom{n-a}{k-a}\frac{\binom{n}{k}}{\binom{2n}{2k}}=2^{2n-2a}\frac{\binom{2a}{a}}{\binom{2n}{n}},\qquad a \text{ a nonnegative integer, } a\le n \tag{10.22}$$

$$\sum_{k=1}^{n}\binom{n}{k}\frac{(yk)^{n-k}(x-yk)^k}{k}=\frac{x^n}{n},\qquad n\ge 2 \tag{10.23}$$

$$\sum_{k=1}^{n}\binom{n}{k}(yk)^{n-k}(x-yk)^{k-1}=x^{n-1},\qquad n\ge 1 \tag{10.24}$$

$$\sum_{k=1}^{n}\binom{n}{k}k^{n-k}(x-k)^{k-1}=x^{n-1},\qquad n\ge 1 \tag{10.25}$$

Application of Equation (10.24)

$$\sum_{j=0}^{n}a_jx^j=f(x)=\sum_{k=0}^{n}(x+yk+y)^k\sum_{j=k}^{n}\binom{j+1}{k+1}(-yk-y)^{j-k}a_j \tag{10.26}$$

Abel's Sum

$$(x+y+nz)^n=x\sum_{j=0}^{n}\binom{n}{j}(y+(n-j)z)^{n-j}(x+jz)^{j-1} \tag{10.27}$$

Application of Equation (10.24) to an Expansion of e^{xz}

$$e^{xz}=x\sum_{k=0}^{\infty}\frac{(x+yk)^{k-1}z^ke^{-ykz}}{k!} \tag{10.28}$$

Remark 10.5 *The following identity is M. S. Klamkin's solution to Problem 4489 [1952, 332] of The American Math. Monthly, Sept. 1953, P. 485.*

$$\sum_{k=1}^{\infty} \frac{((x+yk)w)^{k-1}}{k!} = \frac{e^{xz}-1}{wx}, \qquad |w| < \frac{1}{|y|e},\ x \neq 0 \tag{10.29}$$

$$\sum_{k=1}^{\infty} \frac{(ywk)^{k-1}}{k!} = \frac{z}{w}, \qquad w = ze^{-yz} \tag{10.30}$$

$$\sum_{k=1}^{\infty} \frac{(xe^{-x})^{k-1}k^{k-1}}{k!} = e^{x} \tag{10.31}$$

$$\sum_{k=1}^{\infty} \frac{\left(\frac{k}{e}\right)^{k-1}}{k!} = e \tag{10.32}$$

$$\sum_{k=1}^{\infty} \frac{k^k}{k!}(xe^{-x})^{k-1} = \frac{e^x}{1-x}, \qquad x \neq 1 \tag{10.33}$$

Extension of Equation (10.32)

$$\sum_{k=0}^{\infty} \frac{x^{k-1}}{\binom{x+k}{k}k!} = \sum_{k=1}^{\infty} \frac{k^k}{k!}\frac{e^{-k}}{x+k} \tag{10.34}$$

Application of Equation (10.33)

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}k^{n-k}(k+1)^k = n!\sum_{j=0}^{n}\frac{1}{j!} \tag{10.35}$$

Three Applications of Equation (10.35)

Remark 10.6 *The following identity is from Problem E1318 of The American Math, Monthly, Vol. 65, 1958, P. 366. Recall that for real x, $[x]$ denotes the floor of x.*

$$\sum_{k=0}^{n} \frac{1}{k!} = \frac{[en!]}{n!}, \tag{10.36}$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} [ek!] = (-1)^n n! + 1 \tag{10.37}$$

$$\sum_{k=1}^{n} (-1)^k \binom{n}{k} k^n \left(1 + \frac{1}{k}\right)^k = (-1)^n [en!], \qquad n \geq 1 \tag{10.38}$$

$$\sum_{k=1}^{\infty} \frac{1}{\binom{n+k}{k} k!} = en! - [en!], \qquad n \geq 1 \tag{10.39}$$

$$\sum_{k=1}^{n} k[ek!] = [e(n+1)!] - n - 2, \qquad n \geq 1 \tag{10.40}$$

11 Newton-Gregory Expansions for Polynomials

Remark 11.1 *Throughout this chapter, we will assume $f(x)$ is a polynomial of degree n, namely, $f(x) = \sum_{i=0}^{n} a_i x^i$. We will also assume, unless otherwise specified, that x and y are real or complex numbers, while n is a nonnegative integer.*

Newton-Gregory Expansions

$$\boldsymbol{f(x+y) = \sum_{k=0}^{n} \binom{x}{k} \Delta_1^k f(z)|_{z=y}} \tag{11.1}$$

$$\boldsymbol{f(x+y) = \sum_{k=0}^{n} \binom{\frac{x}{h}}{k} \Delta_h^k f(y),} \qquad \text{where } \boldsymbol{h} \text{ is a nonzero real or complex number} \tag{11.2}$$

12 Recursive Formula for Functions Defined by n^{th} Differences

Remark 12.1 *Throughout this chapter, we will assume n is a nonnegative integer. We also let, for real x, $[x]$ denote the floor of x.*

12.1 First Recursive Formula

If

$$F(n) = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \binom{n-k}{k} \frac{f(k)}{n-k}, \tag{12.1}$$

then

$$\sum_{k=0}^{[\frac{n}{2}]} (-1)^k \binom{n-k}{k} \frac{f(k+1)}{k+1} = \frac{f(0)}{n+2} - F(n+2). \tag{12.2}$$

Note that $f(k)$ independent of n.

12.2 Second Recursive Formula

If

$$F(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(k), \tag{12.3}$$

then

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{f(k+1)}{k+1} = \frac{f(0) - F(n+1)}{n+1}. \tag{12.4}$$

Note that $f(k)$ independent of n.

12.3 Third Recursive Formula

Remark 12.2 *Throughout this section, we assume r and α are integers.*

Define

$$F_n(r+1) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{f(k+r+1)}{\binom{k+r+1}{k}}, \qquad r \geq -1. \tag{12.5}$$

Then,

$$F_n(r+1) = \frac{r+1}{n+1} \left(f(r) - \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} \frac{f(k+r)}{\binom{k+r}{k}} \right). \tag{12.6}$$

Also,

$$\begin{aligned} F_n(\alpha+1) &= (-1)^\alpha \frac{\alpha+1}{\binom{n+\alpha}{\alpha}} \left(F_{n+\alpha}(1) - \sum_{j=0}^{\alpha-1} (-1)^j \binom{n+\alpha}{j} \frac{f(j+1)}{j+1} \right) \\ &= (-1)^\alpha \frac{\alpha+1}{\binom{n+\alpha}{\alpha}} \sum_{j=\alpha}^{n+\alpha} (-1)^j \binom{n+\alpha}{j} \frac{f(j+1)}{j+1}, \qquad \alpha \geq 1. \end{aligned} \tag{12.7}$$

12.3.1 Applicati0ns of Equation (12.7)

$$F_n(n+1) = (-1)^n \frac{n+1}{\binom{2n}{n}} \sum_{j=n}^{2n} (-1)^j \binom{2n}{j} \frac{f(j+1)}{j+1} \tag{12.8}$$

Remark 12.3 *The following identity, due to H. F. Sandham, is from Advanced Problem 4519 of The American Math. Monthly, January 1953, Vol. 60, No. 1, P. 47.*

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{(2k+1)\binom{k+n+1}{k}} = \frac{1}{4(n+1)} \left(\frac{(n+1)!^2 \cdot 2^{2n+2}}{(2n+2)!} \right)^2 \tag{12.9}$$

$$\sum_{k=0}^{2n+1} (-1)^k \binom{2n+1}{k} \frac{1}{2n-2k+1} = (-1)^n \frac{2^{4n+1}}{(2n+1)\binom{2n}{n}} \tag{12.10}$$

$$\sum_{k=0}^{n} (-1)^k \binom{2n+1}{k} \frac{1}{2n-2k+1} = (-1)^n \frac{2^{4n}}{(2n+1)\binom{2n}{n}} \tag{12.11}$$

$$\sum_{k=0}^{n} (-1)^k \binom{n+\alpha}{n-k} \binom{k+\alpha+1}{k} = \begin{cases} 1, & n=0 \\ -1, & n=1 \\ 0, & n \geq 2 \end{cases} \quad \alpha \text{ is a real or complex number} \tag{12.12}$$