Identities for Fibonacci and Lucas Polynomials derived from a book of Gould

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Abstract

This note is dedicated to Professor Gould. The aim is to show how the identities in his book "Combinatorial Identities" can be used to obtain identities for Fibonacci and Lucas polynomials. In turn these identities allow to derive a wealth of numerical identities for Fibonacci and Lucas numbers.

1 Introduction

The book of Gould [3] is an almost endless source of applications in many fields: in this note we give examples showing how identities for Fibonacci and Lucas polynomials can be derived. As a consequence many numerical identities can be obtained choosing appropriate numerical values for the variables x and y.

We define bivariate Fibonacci polynomials as

$$F_n(x,\, y) = xF_{n-1}(x,\, y) + yF_{n-2}(x,\, y), \qquad F_0(x,\, y) = 0,\, F_1(x,\, y) = 1,$$

and bivariate Lucas polynomials as

$$L_n(x,\, y) = xL_{n-1}(x,\, y) + yL_{n-2}(x,\, y), \qquad L_0(x,\, y) = 2,\, L_1(x,\, y) = x.$$

We assume $x \neq 0$, $y \neq 0$, $x^2 + 4y \neq 0$.
The roots of the characteristic equation are

$$\alpha \equiv \alpha(x,\, y) = \frac{x + \sqrt{x^2 + 4y}}{2}, \qquad \beta \equiv \beta(x,\, y) = \frac{x - \sqrt{x^2 + 4y}}{2}.$$

We have $\alpha+\beta = x$, $\alpha\beta = -y$ and $\alpha - \beta = \sqrt{x^2+4y}$. The Binet's forms are

$$F_n(x,\,y) = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad L_n(x,\,y) = \alpha^n + \beta^n.$$

The generating function of the Fibonacci polynomials is

$$\frac{t}{1-xt-yt^2},$$

that of the Lucas polynomials is

$$\frac{2-xt}{1-xt-yt^2}.$$

Many basic facts concerning these kinds of polynomials can be found in [1], [2].

2 Examples

All the references are to the book [3].

1. Identity 1.64 says

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} \frac{1}{n-k} \left(\frac{z}{4}\right)^k = \frac{1}{n2^{n-1}} \frac{u^n+v^n}{u+v},$$

with $u = 1+\sqrt{z+1}$, $v = 1-\sqrt{z+1}$.
Write $z = 4\frac{y}{x^2}$. Then

$$u = 1 + \sqrt{1+4\frac{y}{x^2}} = \frac{x+\sqrt{x^2+4y}}{x},$$

from which $u = \frac{2\alpha}{x}$. In the same way $v = \frac{x-\sqrt{x^2+4y}}{x} = \frac{2\beta}{x}$. The RHS of the previous identity becomes

$$\frac{L_n(x,\,y)}{nx^n},$$

so that we get the identity

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} \frac{n}{n-k} x^{n-2k} y^k = L_n(x,\,y).$$

2. Identity 1.38 says

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} \frac{z^{2k}}{2k+1} = \frac{(1+z)^{n+1} - (1-z)^{n+1}}{2(n+1)z}.$$

Write $z = \frac{\alpha-\beta}{x}$. Then $1+z = \frac{2\alpha}{x}$, $1-z = \frac{2\beta}{x}$. The RHS of the identity becomes

$$\frac{2^n}{(n+1)x^n} F_{n+1}(x,\, y),$$

so that we get the identity

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} \frac{(x^2+4y)^k x^{n-2k}}{2k+1} = \frac{2^n}{(n+1)} F_{n+1}(x,\, y).$$

3. Identity 1.39 says

$$\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1} \frac{z^{2k}}{k+1} = \frac{(1+z)^{n+1} + (1-z)^{n+1} - 2}{(n+1)z^2}.$$

Multiply both sides by z^2 and again write $z = \frac{\alpha-\beta}{x}$. The identity becomes

$$\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1} \frac{(x^2+4y)^{k+1}}{(k+1)x^{2k+2}} = \frac{2^{n+1} L_{n+1}(x,\, y) - 2x^{n+1}}{(n+1)x^{n+1}},$$

that is

$$(n+1) \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1} \frac{(x^2+4y)^{k+1} x^{n-2k-1}}{k+1} = 2^{n+1} L_{n+1}(x,\, y) - 2x^{n+1}.$$

4. Identity 1.61 says

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} 2^{n-2k} y^k = \frac{\left(1+\sqrt{1+y}\right)^{n+1} - \left(1-\sqrt{1+y}\right)^{n+1}}{2\sqrt{1+y}}.$$

Consider the polynomials $F_n(2,\, y)$. The roots of the characteristic equation are

$$\alpha = 1 + \sqrt{1+y}, \qquad \beta = 1 - \sqrt{1+y}.$$

It follows

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} 2^{n-2k} y^k = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} = F_{n+1}(2,\, y).$$

Recall that $F_n(2,\, 1)$ are the Pell numbers P_n.

5a. Identity 1.87 says

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} z^k = \frac{(1+\sqrt{z})^n + (1-\sqrt{z})^n}{2}.$$

With $z = 1 + 4\frac{y}{x^2}$ we have

$$\begin{aligned} 1+\sqrt{z} &= \frac{x+\sqrt{x^2+4y}}{x} = \frac{2\alpha}{x}, \\ 1-\sqrt{z} &= \frac{x-\sqrt{x^2+4y}}{x} = \frac{2\beta}{x}. \end{aligned}$$

Then

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} \left(\frac{x^2+4y}{x^2}\right)^k = \frac{2^{n-1}}{x^n} L_n(x,\, y),$$

that is

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} (x^2+4y)^k x^{n-2k} = 2^{n-1} L_n(x,\, y).$$

5b. In the same Identity 1.87 replace z by $\frac{\alpha^2}{x^2}$ so that

$$1+\sqrt{z} = \frac{x+\alpha}{x}, \qquad 1-\sqrt{z} = \frac{\beta}{x},$$

and then by $\frac{\beta^2}{x^2}$ so that now

$$1+\sqrt{z} = \frac{x+\beta}{x}, \qquad 1-\sqrt{z} = \frac{\alpha}{x}.$$

After summation, the identity becomes

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} \frac{L_{2k}(x,\, y)}{x^{2k}} = \frac{1}{2x^n} L_n(x,\, y) + \frac{1}{2}\left[\left(\frac{x+\alpha}{x}\right)^n + \left(\frac{x+\beta}{x}\right)^n\right].$$

Let

$$\tilde{\alpha} = x + \alpha = \frac{3x + \sqrt{x^2 + 4y}}{2},$$

$$\tilde{\beta} = x + \beta = \frac{3x - \sqrt{x^2 + 4y}}{2}.$$

Then $\tilde{\alpha}$ and $\tilde{\beta}$ are the roots of the characteristic equation of $L_n(3x,\, y - 2x^2)$ as it can be checked easily. Then

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} \frac{L_{2k}(x,\, y)}{x^{2k}} = \frac{1}{2x^n} \left[L_n(x,\, y) + L_n(3x,\, y - 2x^2) \right],$$

that is

$$2 \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} L_{2k}(x,\, y) x^{n-2k} = L_n(x,\, y) + L_n(3x,\, y - 2x^2).$$

6a. Identity 1.95 says

$$\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1} z^k = \frac{(1 + \sqrt{z})^n - (1 - \sqrt{z})^n}{2\sqrt{z}}.$$

As in Example 5a, write $z = 1 + 4\frac{y}{x^2}$. Then

$$\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1} (x^2 + 4y)^k x^{n-2k-1} = 2^{n-1} F_n(x,\, y).$$

6b. In the same Identity 1.95 replace, as in Example 5b, z a first time by $\frac{\alpha^2}{x^2}$ and then by $\frac{\beta^2}{x^2}$. Subtract the resulting identities and divide by $\alpha - \beta$. The LHS is now

$$\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1} \frac{F_{2k}(x,\, y)}{x^{2k}}.$$

The RHS after some simple manipulations becomes

$$\frac{1}{2x^{n-1}(\alpha - \beta)} \left[\frac{(x + \alpha)^n - \beta^n}{\alpha} - \frac{(x + \beta)^n - \alpha^n}{\beta} \right],$$

that is

$$\frac{1}{2x^{n-1}(-y)(\alpha-\beta)}\left[(\alpha-\beta)F_{n+1}(x,\,y)+\beta(x+\alpha)^n-\alpha(x+\beta)^n\right].$$

As in Example 5b, let

$$\tilde{\alpha}=x+\alpha=\frac{3x+\sqrt{x^2+4y}}{2},$$
$$\tilde{\beta}=x+\beta=\frac{3x-\sqrt{x^2+4y}}{2}.$$

Again $\tilde{\alpha}$ and $\tilde{\beta}$ are the roots of the characteristic equation of $L_n(3x,\,y-2x^2)$ or $F_n(3x,\,y-2x^2)$. Then $\tilde{\alpha}-\tilde{\beta}=\alpha-\beta$ and $\tilde{\alpha}\tilde{\beta}=-y+2x^2$. It follows

$$\begin{aligned}\beta(x+\alpha)^n &= (\tilde{\beta}-x)\tilde{\alpha}^n\\ &= \tilde{\beta}\tilde{\alpha}^n-x\tilde{\alpha}^n\\ &= \tilde{\beta}\tilde{\alpha}\tilde{\alpha}^{n-1}-x\tilde{\alpha}^n\\ &= (-y+2x^2)\tilde{\alpha}^{n-1}-x\tilde{\alpha}^n.\end{aligned}$$

In the same way

$$\alpha(x+\beta)^n=(-y+2x^2)\tilde{\beta}^{n-1}-x\tilde{\beta}^n.$$

So the RHS now becomes

$$\frac{1}{2x^{n-1}(-y)}\Big[F_{n+1}(x,\,y)-(y-2x^2)F_{n-1}(3x,\,y-2x^2)-xF_n(3x,\,y-2x^2)\Big].$$

Finally we get the identity

$$\begin{aligned}&2y\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n}{2k+1}F_{2k}(x,\,y)x^{n-2k-1}\\ &\quad=-\Big[F_{n+1}(x,\,y)-(y-2x^2)F_{n-1}(3x,\,y-2x^2)-xF_n(3x,\,y-2x^2)\Big].\end{aligned}$$

Remark. $F_n(1,\,1)$ and $L_n(1,\,1)$ are, respectively, the Fibonacci and Lucas numbers. In the above Examples, with the exception of 4, 5b, 6b, all the identities can be reduced to identities involving Fibonacci and Lucas numbers by setting $x=L_k(1,\,1)$, $y=(-1)^{k+1}$. This is due to a consequence of a result in [2]:

$$F_n\left(L_k(x,\,y),\,(-1)^{k+1}y^k\right)=\frac{F_{kn}(x,\,y)}{F_k(x,\,y)},$$
$$L_n\left(L_k(x,\,y),\,(-1)^{k+1}y^k\right)=L_{kn}(x,\,y).$$

References

[1] M. Catalani (2004), "Generalized Bivariate Fibonacci Polynomials." Version 2. http://front.math.ucdavis.edu/math.CO/0211366

[2] M. Catalani (2004), "Some Formulae for Bivariate Fibonacci and Lucas Polynomials." http://front.math.ucdavis.edu/math.CO/0406323

[3] H.W. Gould (1972), *Combinatorial Identities*, Morgantown, W. Va.