Determine the number of odd binomial coefficients in the expansion of $(x+y)^{1000}$.

Theorem 0.1. *The number of odd entries in row n of Pascal's Triangle is 2 raised to the number of 1's in the binary expansion of n.*

Proof. The binomial theorem says that

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

So with $a = 1$ and $b = x$ we have

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

If we reduce the coefficients mod 2, then it's easy to show by induction on n that for $n \geq 0$,

$$(1+x)^{2^n} \equiv (1+x^{2^n}) (\text{mod } 2).$$

Thus:

$$(1+x)^{10} = (1+x)^8(1+x)^2 \equiv (1+x^8)(1+x^2)(\text{mod } 2) = (1+x^2+x^8+x^{10})(\text{mod } 2).$$

Since the coefficients of these polynomials are equal mod 2, using the binomial theorem we see that $\binom{n}{k}$ is odd for $k = 0, 2, 8, 10$, and it is even for all other k. Similarly, the product

$$(1+x)^{11} \equiv (1+x^8)(1+x^2)(1+x^1)(\text{mod } 2)$$

is a polynomial containing $8 = 2^3$ terms, being the product of 3 factors with 2 choices in each.

In general, if n can be expressed as the sum of p distinct powers of 2, then $\binom{n}{k}$ will be odd for 2^p values of k. But p is just the number of 1's in the binary expansion of n, and $\binom{n}{k}$ are the numbers in the n^{th} row of Pascal's triangle. $\square$