Three-variable inequalities

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1 Theorem

For any triad of numbers a, b, c we denote that:

$$\begin{aligned} p &= a+b+c \\ q &= ab+bc+ca \\ r &= abc \end{aligned}$$

We have the following identities:

$$\begin{aligned} a^2+b^2+c^2 &= p^2-2q \\ a^3+b^3+c^3 &= p^3-3pq+3r \\ a^2(b+c)+b^2(c+a)+c^2(a+b) &= pq-3r \\ a^4+b^4+c^4 &= p^4+2q^2+4pr-4p^2q \\ a^2b^2+b^2c^2+c^2a^2 &= q^2-2pr \\ a^3(b+c)+b^3(c+a)+c^3(a+b) &= p^2q-2q^2-pr \\ (a-b)^2(b-c)^2(c-a)^2 &= p^2q^2+18pqr-27r^2-4q^3-4p^3r \end{aligned}$$

The expression $f(X) = AX^2+BX+C$: $\begin{cases} A & \geq & 0 \\ X_{min} & = & \dfrac{-B}{2A} \end{cases}$

(+) $\quad f(X) \geq 0\,\forall X \Leftrightarrow \Delta = B^2-4AC \leq 0$

$$(+)\quad f(X)\geq 0\,\forall X\geq 0\Leftrightarrow \left[\begin{array}{l}\left\{\begin{array}{lcl}X_{min} & \leq & 0\\ f(0) & \geq & 0\end{array}\right.\\ \left\{\begin{array}{lcl}X_{min} & \geq & 0\\ f(X_{min}) & \geq & 0\end{array}\right.\end{array}\right. \Leftrightarrow \left[\begin{array}{l}\left\{\begin{array}{lcl}B & \geq & 0\\ C & \geq & 0\end{array}\right.\\ \left\{\begin{array}{lcl}B & \leq & 0\\ \Delta = B^2-4AC & \geq & 0\end{array}\right.\end{array}\right.$$

2 Applications

We have already known the applications of the method using p, q, r in the proof for symetric inequalities, and here are some applications of this for cyclic inequalities, note that some problem are very hard.

Example 1: Let a, b, c be non-negative real numbers satisfying $a+b+c=3$. Prove that:

$$a^2b+b^2c+c^2a\leq 4 \qquad (*)$$

SOLUTION.
We have:

$$(*)\Leftrightarrow 2\sum_{cyc}a^2b\leq 8$$

$$\Leftrightarrow \sum_{cyc}a^2b-\sum_{cyc}ab^2\leq 8-\sum_{cyc}a^2b-\sum_{cyc}ab^2$$

$$\Leftrightarrow (a-b)(b-c)(a-c)\leq 8-\sum_{sym}a^2(b+c)$$

We only need to prove when $(a-b)(b-c)(a-c)\geq 0$, so the inequality is equivalent to:

$$\sqrt{(a-b)^2(b-c)^2(c-a)^2}\leq 8-\sum_{sym}a^2(b+c)$$

$$\Leftrightarrow \sqrt{(p^2q^2+18pqr-27r^2-4q^3-4q^3r)}\leq 8-(pq-3r)$$

$$\Leftrightarrow (p^2q^2+18pqr-27r^2-4q^3-4p^3r)\leq (8-pq+3r)^2$$

$$\Leftrightarrow 36r^2+(4p^3-24pq+48)r+4q^3-16pq+64\geq 0$$

$$\Leftrightarrow f(r)=9r^2+(p^3-6pq+12)r+q^3-4pq+16\geq 0$$

For $p=3$:

$$f(r)=9r^2+(39-18q)r+q^3-12q+16\geq 0$$

Observe that $r_{min}=\dfrac{-39+18q}{18}$. Consider two cases:
If $0\leq q\leq\dfrac{39}{18}\Rightarrow r_{ct}\leq 0$, then:

$$f(0)=q^3-12q+16=(q+4)(q-2)^2\geq 0$$

If $\dfrac{39}{18} \leq q \leq 3 \Rightarrow r_{min} \geq 0$, we have:

$$f(r_{min}) = 24q^3 - 216q^2 + 648q - 630 \geq 0 \, \forall q \in \left[\frac{39}{18}; 3\right]$$

Therefore we have $f(r) \geq 0 \, \forall r \geq 0$, the inequality is proved.

Example 2: Let a, b, c be non-negative real numbers adding up to 3. Prove that:

$$a^2b + b^2c + c^2a + 2(ab^2 + bc^2 + ca^2) \leq 6\sqrt{3}$$

SOLUTION.
The inequality is equivalent to:

$$2\sum_{cyc} a^2b + 4\sum_{cyc} ab^2 \leq 12\sqrt{3}$$

$$\Leftrightarrow 3\sum_{sym} a^2(b+c) + \sum_{cyc} ab^2 - \sum_{cyc} a^2b \leq 12\sqrt{3}$$

$$\Leftrightarrow 3\sum_{cyc} a^2(b+c) + (a-b)(b-c)(c-a) \leq 12\sqrt{3}$$

We need to prove the above inequality when $(a-b)(b-c)(c-a) \geq 0$, which is

$$3\sum_{sym} a^2(b+c) + \sqrt{(a-b)^2(b-c)^2(c-a)^2} \leq 12\sqrt{3}$$

$$\Leftrightarrow 3(pq - 3r) + \sqrt{p^2q^2 + 18pqr - 27r^2 - 4q^3 - 4p^3r} \leq 12\sqrt{3}$$

$$\Leftrightarrow p^2q^2 + 18pqr - 27r^2 - 4q^3 - 4p^3r \leq \left(12\sqrt{3} - 3pq + 9r\right)^2$$

$$\Leftrightarrow f(r) = 108r^2 + \left(4p^3 - 72pq + 216\sqrt{3}\right)r + 4q^3 + 8p^2q^2 - 72\sqrt{3}pq + 432 \geq 0$$

Find that $r_{min} = \dfrac{216q - 108 - 216\sqrt{3}}{108}$, consider two cases:
If $0 \leq q \leq \dfrac{216\sqrt{3} + 108}{216} \Rightarrow r_{min} \leq 0$, then:

$$f(0) = 4\left(q + 12 + 6\sqrt{3}\right)\left(q + 3 - \sqrt{3}\right)^2 \geq 0$$

If $\dfrac{216\sqrt{3} + 108}{216} \leq q \leq 3 \Rightarrow r_{ct} \geq 0$, we have:

$$f(r_{ct}) = 4q^3 - 36q^2 + 108q + 81 - 108\sqrt{3} \geq 0 \, \forall q \in \left[\sqrt{3} + \frac{1}{2}; 3\right]$$

Thus $f(r)$ is non-negative for all $r \geq 0$, the inequality is proved.

Example 3: (Pham Sinh Tan). Find the greatest constant k such that the following inequality holds for any non-negative real numbers a, b, c:

$$k(a+b+c)^4 \geq (a^3b+b^3c+c^3a)+(a^2b^2+b^2c^2+c^2a^2)+abc(a+b+c)$$

SOLUTION.
For $a=2, b=1, c=0$ we obtain $k \geq \dfrac{4}{27}$. We will prove that this is the desired value, it means that the given inequality holds for $k=\dfrac{4}{27}$. Without loss of generality, assume that $p=1$, we have:

$$\frac{4}{27}(a+b+c)^4 \geq \sum_{cyc} a^3b + \sum_{sym} b^2c^2 + abc\sum_{sym} a$$

$$\Leftrightarrow \frac{8}{27}(a+b+c)^4 \geq \left(\sum_{cyc} a^3b + \sum_{cyc} ab^3\right) + 2\sum_{sym} b^2c^2 + \left(\sum_{cyc} a^3b - \sum_{cyc} ab^3\right) + 2abc(a+b+c)$$

$$\Leftrightarrow \frac{8}{27}(a+b+c)^4 \geq \sum_{sym} a^3(b+c) + 2\sum_{sym} b^2c^2 + (a+b+c)(a-b)(b-c)(a-c) + 2abc(a+b+c)$$

We only need to consider the case $(a-b)(b-c)(a-c) \geq 0$, then the inequality is equivalent to:

$$\Leftrightarrow \frac{8}{27}(a+b+c)^4 \geq p^2q - 2q^2 - pr + 2q^2 - 4pr + 2pr + p\sqrt{p^2q^2 + 18pqr - 27r^2 - 4q^3 - 4p^3r}$$

$$\Leftrightarrow p^2(p^2q^2 + 18pqr - 27r^2 - 4q^3 - 4p^3r) \leq \left(\frac{8}{27}p^4 - p^2q + 3pr\right)^2$$

$$\Leftrightarrow f(r) = 36p^2r^2 + \left(\frac{52}{9}p^5 - 24p^3q\right)r + \frac{64}{729}p^3 + 4p^2q^3 - \frac{16}{27}p^6q \geq 0$$

For $p=3$:

$$f(r) = 324r^2 + (1404 - 648q)r + 36q^3 - 432q + 576 \geq 0$$

Consider two cases:
If $0 \leq q \leq \dfrac{13}{6} \rightarrow 39 - 18q \geq 0$, then

$$f(0) = 36(q+4)(q-2)^2 \geq 0$$

If $\dfrac{13}{6} \leq q \leq 3$ we have:

$$\Delta = (39-18q)^2 - 4.9.(q^3 - 12q + 16) = -36q^3 + 324q^2 - 972q + 945 \leq 0 \text{ for } q \in \left[\frac{13}{6}; 3\right]$$

Therefore, $f(r) \geq 0 \, \forall r \geq 0$, the inequality is proved.

Example 4: (Varsile Cirtoaje). Prove the following inequality for all real numbers a, b, c:

$$(x^2+y^2+z^2)^2 \geq 3(x^3y+y^3z+z^3x) \qquad (*)$$

SOLUTION.
If $p=0$ the inequality can be rewritten as:

$$7(y^2+z^2+yz)^2 \geq 0$$

which is obviously true.
Consider the case $p \neq 0$, without loss of generality, suppose that $p=3$, we have:

$$(*) \Leftrightarrow 2(a^2+b^2+c^2)^2 \geq 3\left(\sum_{cyc} a^3b + \sum_{cyc} ab^3\right) + 3\left(\sum_{cyc} a^3b - \sum_{cyc} ab^3\right)$$

$$\Leftrightarrow 2(a^2+b^2+c^2)^2 \geq 3\sum_{sym} a^3(b+c) + 3(a+b+c)(a-b)(b-c)(a-c)$$

We only need to prove when $(a-b)(b-c)(a-c) \geq 0$, then the inequality is equivalent to:

$$\Leftrightarrow 2(a^2+b^2+c^2)^2 \geq 3\sum_{sym} a^3(b+c) + 3(a+b+c)\sqrt{(a-b)^2(b-c)^2(c-a)^2}$$

$$\Leftrightarrow 2(p^2-2q)^2 \geq 3(p^2q-2q^2-pr) + 3p\sqrt{p^2q^2+18pqr-27r^2-4q^3-4p^3r}$$

$$\Leftrightarrow 9p^2(p^2q^2+18pqr-27r^2-4q^3-4p^3r) \leq \left[2(p^2-2q)^2-3(p^2q-2q^2-pr)\right]^2$$

$$\Leftrightarrow f(r)252p^2r^2 + \left(84pq^2-228p^3q+48p^5\right)r + 4p^8-44p^4q+168p^4q^2-272p^2q^3+196q^4 \geq 0$$

For $p=3$:

$$f(r) = 4\left(567r^2+(63q^2-1539q+2916)r+49q^4-612q^3+3402q^2-8019q+6561\right) \geq 0$$

We have:

$$\Delta' = (63q^2-1539q+2916)^2-2268(49q^4-612q^3+3402q^2-8019q+6561) = -2187(7q-18)^2(q-3)^2 \leq 0$$

Hence $f(r) \geq 0$ for all real number r, this ends the proof.

Example 5: Determine the greatest constant k such that the following inequality holds for any positive real numbers a, b, c:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+k\frac{ab+bc+ca}{a^2+b^2+c^2} \geq 3+k$$

SOLUTION.

The inequality is equivalent to the following one:

$$\left(\sum_{cyc}\frac{a}{b}+\sum_{cyc}\frac{b}{a}\right)+2k\frac{ab+bc+ca}{a^2+b^2+c^2}\geq 6+2k-\left(\sum_{cyc}\frac{b}{a}-\sum_{cyc}\frac{a}{b}\right)$$

$$\Leftrightarrow \frac{\sum\limits_{sym} a^2(b+c)}{abc}+2k\frac{ab+bc+ca}{a^2+b^2+c^2}\geq 6+2k+\frac{(a-b)(b-c)(a-c)}{abc}$$

We only need to prove when $(a-b)(b-c)(a-c)\geq 0$

$$\Leftrightarrow \frac{pq-3r}{r}+\frac{2kq}{p^2-2q}\geq 6+2k+\frac{\sqrt{p^2q^2+18pqr-27r^2-4q^3-4p^3r}}{r}$$

$$\left(p^2q^2+18pqr-27r^2-4q^3-4p^3q\right)(p^2-2q)^2\leq \left[(pq-3r)(p^2-2q)+2kqr-(6+2k)r(p^2-2q)\right]^2$$

Without loss of generality, assume that $p=3$. After expanding, the above inequality is written as:

$$f(r)=4(Ar^2+Br+C)\geq 0$$

For:

$$\begin{aligned}
A &= 81k^2+9k^2q^2+54kq^2+729k-972q+108q^2+2187-54k^2q-405kq\\
B &= 2187-108q^3+1080q^2-3159q-18kq^3-243kq+135kq^2\\
C &= 4q^5-36q^4+81q^3
\end{aligned}$$

It is easy to prove that A and C is non-negative. Consider two cases:

If $0\leq q\leq \dfrac{3\left(k+11-\sqrt{k^2+10k+49}\right)}{2(k+6)}\Rightarrow B\geq 0\Rightarrow f(r)\geq 0$

If $\dfrac{3\left(k+11-\sqrt{k^2+10k+49}\right)}{2(k+6)}\leq q\leq 3$ then

$$\Delta=B^2-4AC=-9(q-3)^2(2q-9)^2\left(48q^3+24kq^3+4k^2q^3-144kq^2-468q^2-9k^2q^2+162kq+1296q-719\right)$$

whence we can find that $k_{max}=3\sqrt[3]{4}-2$, this is the desired value.

Example 6: (Bach Ngoc Thanh Cong). Find the greatest constant k such that the following inequality holds for all positive real numbers a,b,c:

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}+k(a+b+c)\geq 3(k+1)\frac{a^2+b^2+c^2}{a+b+c}\qquad (*)$$

SOLUTION.

We observe that:

$$2\sum_{cyc}\frac{a^2}{b}=\left(\sum_{cyc}\frac{a^2}{b}+\sum_{cyc}\frac{b^2}{a}\right)+\left(\sum_{cyc}\frac{a^2}{b}-\sum_{cyc}\frac{b^2}{a}\right)=\frac{\sum\limits_{cyc}a^3(b+c)}{abc}+\frac{(a+b+c)(a-b)(b-c)(c-a)}{abc}$$

Hence

$$(*)\Leftrightarrow 2\sum_{cyc}\frac{a^2}{b}+2k(a+b+c)\geq 6(k+1)\frac{a^2+b^2+c^2}{a+b+c}$$

$$\Leftrightarrow \frac{\sum\limits_{cyc}a^3(b+c)}{abc}+2k(a+b+c)-6(k+1)\frac{a^2+b^2+c^2}{a+b+c}\geq\frac{(a+b+c)(a-b)(b-c)(a-c)}{abc}$$

Consider the case $(a-b)(b-c)(a-c)\geq 0$, the above inequality can be rewritten as:

$$\frac{p^2q-2q^2-pr}{r}+2kp-6(k+1)\frac{p^2-2q}{p}\geq\frac{p\sqrt{p^2q^2+18pqr-27r^2-4q^3-4p^3r}}{r}$$

$$\Leftrightarrow f(r)=\left[(p^2q-2q^2-pr)p+2kp^2r-6(k+1)r(p^2-2q)\right]^2-p^4(p^2q^2+18pqr-27r^2-4q^3-4p^3r)\geq 0$$

Similarly, suppose that $p=3$, after expanding we have:

$$f(r)=4\left(Ar^2+Br+C\right)\geq 0$$

In which:

$$\begin{aligned}A&=72kq^2+36k^2q^2+324k^2-378q+1134k-594kq-216k^2q+36q^2+1539\\ B&=2187-486kq+270kq^2-36kq^3-1944q+351q^2-36q^3\\ C&=9q^4\end{aligned}$$

The root q_o of the equation $B=0$ satisfying $q_o\in[0,3]$ is:

$$q_o=\frac{1}{4(1+k)}\left(\sqrt[3]{M}+\frac{28k^2-100k-119}{\sqrt[3]{M}}+10k+13\right)$$

For:

$$\begin{aligned}M&=-1475-2382k-960k^2-80k^3+36\sqrt{N}+36k\sqrt{N}\\ N&=-12k^4+324k63-63k^2+2742k+2979\end{aligned}$$

Consider two cases:
If $0\leq q\leq q_o$ then $B\geq 0$, we can prove that A and C are non-negative, thus $f(r)\geq 0$
If $q_o\leq q\leq 3$ then

$$\Delta=B^2-4AC=-729(q-3)^2\left(16q^3+16k^2q^3+32kq^3-252kq^2-189q^2-36k^2q^2+324kq+810q-729\right)$$

Now we can find that $k_{max}\approx 1,5855400068$, this is the desired value.

Example 7: (Bach Ngoc Thanh Cong - Nguyen Vu Tuan). Determine the greatest constant k such that the following inequality is true for all positive real number a, b, c:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+k \geq \frac{(9+3k)(a^2+b^2+c^2)}{(a+b+c)^2}$$

SOLUTION.

The inequality is equivalent to:

$$\left(\sum_{cyc}\frac{a}{b}+\sum_{cyc}\frac{b}{a}\right)+2k \geq \frac{6(3+k)(a^2+b^2+c^2)}{(a+b+c)^2}+\left(\sum_{cyc}\frac{b}{a}-\sum_{cyc}\frac{a}{b}\right)$$

$$\Leftrightarrow \frac{\sum_{sym} a^2(b+c)}{abc}+2k-\frac{6(3+k)(a^2+b^2+c^2)}{(a+b+c)^2} \geq \frac{(a-b)(b-c)(a-c)}{abc}$$

We only need to prove when $(a-b)(b-c)(a-c) \geq 0$, then the above inequality can be rewritten as:

$$\frac{pq-3r}{r}+2k-\frac{6(3+k)(p^2-2q)}{p^2} \geq \frac{\sqrt{p^2q^2+18pqr-27r^2-4q^3-4p^3r}}{r}$$

$$\Leftrightarrow f(r)=\left[(pq-3r)p^2+2kp^2r-6(3+k)(p^2-2q)r\right]^2-(p^2q^2+18pqr-27r^2-4q^3-4p^3r)p^4 \geq 0$$

Suppose that $p=3$, after expanding we have:

$$f(r)=4\left(Ar^2+Br+C\right) \geq 0$$

$$\begin{aligned}
A &= 144k^2q^2+864kq^2+1296k^2-13608q+13608k+1296q^2-864k^2q-7128kq+37098\\
B &= 162kq^2-486kq-3645q+486q^2+2187\\
C &= 81q^3
\end{aligned}$$

Consider two cases:

If $0 \leq q \leq \dfrac{3\left(15+2k-\sqrt{153+36k+4k^2}\right)}{4(3+k)}$ then B is non- negative, similarly to the previous one, we can prove that $A \geq 0, C \geq 0$, we obtain $f(r) \geq 0$

If $\dfrac{3\left(15+2k-\sqrt{153+36k+4k^2}\right)}{4(3+k)} \leq q \leq 3$ we have:

$$\Delta=B^2-4AC=-729(q-3)^2\left(16k^2q^3+96kq^3+144q^3-36k^2q^2-972q^2-432kq^2+324kq+1944q-729\right)$$

Hence we can find that $k_{max}=3\sqrt[3]{2}-3$, this is the desired value, the proof is complete.

Example 8: (Bach Ngoc Thanh Cong). Find the greatest constant k such that the following inequality holds for all $a, b, c > 0$:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq \frac{k(a^2+b^2+c^2)}{ab+bc+ca}-k+3$$

SOLUTION.
After multiplying the inequality by 2 we have:

$$\left(\sum_{cyc}\frac{a}{b}+\sum_{cyc}\frac{b}{a}\right)\geq\frac{2k(a^2+b^2+c^2)}{ab+bc+ca}-2k+6+\left(\sum_{cyc}\frac{b}{a}-\sum_{cyc}\frac{a}{b}\right)$$

$$\Leftrightarrow\frac{\sum\limits_{sym}a^2(b+c)}{abc}-\frac{2(a^2+b^2+c^2)}{ab+bc+ca}+2k-6\geq\frac{(a-b)(b-c)(a-c)}{abc}$$

We only need to prove in the case $(a-b)(b-c)(a-c)\geq 0$, whence the inequality can be rewritten as:

$$\frac{pq-3r}{r}-\frac{2k(p^2-2q)}{q}+2k-6\geq\frac{\sqrt{p^2q^2+18pqr-27r^2-4q^3-4p^3r}}{r}$$

$$\Leftrightarrow f(r)=\left[(pq-3r)q-2kr(p^2-2q)+(2k-6)rq\right]^2-q^2(p^2q^2+18pqr-27r^2-4q^3-4p^3r)\geq 0$$

Assume that $p=3$, by expanding we obtain:

$$f(r)=4\left(Ar^2+Br+C\right)\geq 0$$

where

$$\begin{aligned}A&=9k^2q^2-54k^2q+81k^2-27kq^2+81kq+27q^2\\B&=9kq^3-27q^3+27q^2-27kq^2\\C&=q^5\end{aligned}$$

Thus we can find that $k_{max}=1$, this is the desired value.

Example 9: (Pham Sinh Tan). Find all real number k wuch that the following inequality holds for any real numbers a,b,c:

$$a(a+kb)^3+b(b+kc)^3+c(c+ka)^3\geq\frac{(k+1)^3}{27}(a+b+c)^4\qquad(*)$$

SOLUTION.
If $p=0$ then

$$(*)\Leftrightarrow-(b^2+bc+c^2)^2(k-2)(k^2-k+1)\geq 0$$

Hence the necessary condition is $k \le 2$.

Consider the case $p \neq 0$, without loss of generality, assume that $p = 3$. Observe that:

$$\begin{aligned}
& 6k\sum_{cyc} a^3b + 2k^3\sum_{cyc} ab^3 \\
= \;& 3k\left(\sum_{cyc} a^3b + \sum_{cyc} ab^3\right) + k^3\left(\sum_{cyc} ab^3 + \sum_{cyc} a^3b\right) + 3k\left(\sum_{cyc} a^3b - \sum_{cyc} ab^3\right) + k^3\left(\sum_{cyc} ab^3 - \sum_{cyc} a^3b\right) \\
= \;& (k^3+3k)\sum_{sym} a^3(b+c) + k(k^2-3)(a+b+c)(a-b)(b-c)(c-a)
\end{aligned}$$

Thus we have:

$$(*) \Leftrightarrow 2\sum_{sym} a^4 + 6k\sum_{cyc} a^3b + 2k^3\sum_{cyc} ab^3 + 6k^2\sum_{sym} b^2c^2 \ge \frac{2(k+1)^3}{27}(a+b+c)^4$$

$$\Leftrightarrow 2\sum_{sym} a^4 + (k^3+3k)\sum_{sym} a^3(b+c) + 6k^2\sum_{sym} b^2c^2 - \frac{2(k+1)^3}{27}(a+b+c)^4 \ge k(k^2-3)(a+b+c)(a-b)(b-c)(a-c)$$

We only need to consider the case $RHS \ge 0$, then the inequality can be rewritten as:

$$\begin{aligned}
2(p^4+2q^2+4pr-4p^2q) + 3k(p^2q-2q^2-pr) + k^3(p^2q-2q^2-pr) + 6k^2(q^2-2pr) - \frac{2(k+1)^3p^4}{27} \\
\ge k(k^2-3)p\sqrt{p^2q^2+18pqr-27r^2-4q^3-4p^3r}
\end{aligned}$$

Square two sides of the inequality, then replace p by 3, after expanding the inequality is rewritten as:

$$f(r) = 4\left(Ar^2 + Br + C\right) \ge 0$$

In which:

$$\begin{aligned}
A &= 63k^6 + 54k^5 - 27k^4 + 126k^3 + 135k^2 - 108k + 144 \\
B &= 1872 - 918k + 99k^3 - 756k^2 + 27k^5q^2 + 48q^2 - 864q - 90kq^2 + 51k^3q^2 + 27k^2q^2 - 1080k^4 \\
&\quad +252k^6 + 135k^5 + 648kq - 270k^3q + 81qk^2 - 90k^4q^2 + 648k^4q + 3k^6q^2 - 135k^6q - 162k^5q \\
C &= 6084 - 1476kq^2 - 492k^3q^2 + 486k^2q^2 + 2754kq + 675k^3q + 405qk^2 + 225k^4q^2 - 162k^4q + 6k^6q^2 \\
&\quad -27k^6q - 81k^5q - 1404k - 306k^3 - 1323k^2 + 135k^4 + 9k^6 + 54k^5 - 5616q + 1608q^2 + 27k^5q^3 \\
&\quad -12q^4k - 22q^4k^3 + 21q^4k^2 + 15k^4q^4 + k^6q^4 - 6k^5q^4 - 216k^2q^3 - 108k^4q^3 + 270q^3k + 171q^3k^3 \\
&\quad -144q^3 + 4q^4
\end{aligned}$$

It is easy to prove that A is positive. We have:

$$\begin{aligned}
\Delta = B^2 - 4AC \;=\; & -81k^2(q-3)^2(k^2-3)^2\big[(3k^6 - 18k^5 + 45k^4 - 66k^3 + 63k^2 - 36k + 12)q^2 + (28k^6 \\
& +78k^5 - 174k^4 + 326k^3 - 372k^2 + 276k - 152)q + -84k^6 - 72k^5 + 279k^4 - 168k^3 \\
& +792k^2 + 144k + 780\big]
\end{aligned}$$

whence we can find that the necessary and sufficient condition is:

$$(16k^4-5k^3+30k^2-14k-56)\le 0 \Leftrightarrow -0.9377079399 \le k \le 1.233289162$$

This ends the proof.

Example 10: (Bach Ngoc Thanh Cong-Nguyen Vu Tuan). Determine the greatest constant k such that the following inequality holds for $a,b,c\ge 0$:

$$\frac{a^2+b^2+c^2}{ab+bc+ca}+k\frac{3abc}{ab^2+bc^2+ca^2}\ge 1+k$$

SOLUTION.

The inequality is equivalent to:

$$\begin{aligned}
3kabc+\left(\frac{a^2+b^2+c^2}{ab+bc+ca}-1-k\right)(ab^2+bc^2+ca^2) &\ge 0\\
\Leftrightarrow 6kabc+\left(\frac{a^2+b^2+c^2}{ab+bc+ca}-1-k\right)\left(\sum_{cyc}ab^2+\sum_{cyc}a^2b\right) &\ge \left(\frac{a^2+b^2+c^2}{ab+bc+ca}-1-k\right)\left(\sum_{cyc}a^2b-\sum_{cyc}ab^2\right)
\end{aligned}$$

$$\Leftrightarrow \left(\frac{a^2+b^2+c^2}{ab+bc+ca}-1-k\right)\sum_{sym}a^2(b+c)\ge\left(\frac{a^2+b^2+c^2}{ab+bc+ca}-1-k\right)(a-b)(b-c)(c-a)$$

Consider the case $RHS\ge 0$, then the inequality can be rewritten as:

$$6kr+\left(\frac{p^2-2q}{q}-1-k\right)(pq-3r)\ge\left|\left(\frac{p^2-2q}{q}-1-k\right)\sqrt{p^2q^2+18pqr-27r^2-4q^3-4p^3r}\right|$$

$$\Leftrightarrow f(r)=\left[6kqr+\left(p^2-(k+3)q\right)(pq-3r)\right]^2-\left(p^2-(k+3)q\right)^2\left(p^2q^2+18pqr-27r^2-4q^3-4r\right)\ge 0$$

Without loss of generality, suppose that $p=1$. After expanding we have:

$$f(r)=Ar^2+Br+C$$

For:

$$\begin{aligned}
A &= 108q^2k^2+324q^2k+324q^2-108qk-216q+36\\
B &= -36k^2q^3-180q^3k-216q^3+4q^2k^2+84q^2k+180q^2-8qk-48q+4\\
C &= 4q^3(qk+3q-1)^2
\end{aligned}$$

$$\Delta=-16(3q-1)^2(qk+3q-1)^2(12k^2q^3+36q^3k+36q^3-q^2k^2-24q^2k-36q^2+2qk+12q-1)$$

Now we can find that $k_{max}=k_0\approx 0.8493557485$, this is the desired value.

+) Similar inequality:

$$\frac{3(a^2+b^2+c^2)}{(a+b+c)^2}+k\frac{3abc}{ab^2+bc^2+ca^2}\ge 1+k$$

Example 11: (Bach Ngoc Thanh Cong - Nguyen Vu Tuan). Let a, b, c be positive real numbers. Find the greatest constant k such tthat the following inequality holds:

$$\frac{3(a^2+b^2+c^2)}{(a+b+c)^2}+k\frac{a^3b+b^3c+c^3a}{a^2b^2+b^2c^2+c^2a^2}\geq 1+k$$

SOLUTION. WLOG, assume that $p=1$. Similarly to those previous examples, after changing we only need to prove that:

$$\begin{aligned}
2\frac{3(p^2-2q)}{p^2}+\frac{k(p^2q-2q^2-pr)}{q^2-2pr}-2-2k &\geq \frac{kp\sqrt{p^2q^2+18pqr-27r^2-4q^3-3p^3r}}{q^2-2pr}\\
\Leftrightarrow 2(3-6q)+\frac{k(q-2q^2-r)}{q^2-2r}-2-2k &\geq \frac{k\sqrt{q^2+18qr-27r^2-4q^3-4r}}{q^2-2r}\\
\Leftrightarrow \left[2(3-6q)(q^2-2r)+k(q-2q^2-r)-2(k+1)(q^2-2r)\right]^2 &\geq k^2(q^2+18qr-27r^2-4q^3-4r)\\
\Leftrightarrow f(r)=4(Ar^2+Br+C) &\geq 0
\end{aligned}$$

In which:

$$\begin{aligned}
A &= 144q^2+36qk-96q+9k^2-12k+16\\
B &= -144q^4-66q^3k+96q^3-6q^2k^2+34q^2k-16q^2-3qk^2-4qk+k^2\\
C &= q^3(36q^3+24q^2k-24q^2+4qk^2+2k-14qk+4q-k^2)
\end{aligned}$$

We have:

$$\Delta=-k^2(3q-1)^2(108q^4+72q^3k-80q^3+16q^2-44q^2k+12q^2k^2+8qk-k^2)$$

Hence we can determine that $k_{max}=k_0\approx 1.424183337$.

+)Similar inequality:

$$\frac{3(a^2+b^2+c^2)}{(a+b+c)^2}+k\frac{a^2b^2+b^2c^2+c^2a^2}{a^3b+b^3c+c^3a}\geq 1+k$$

Example 12: (Bach Ngoc Thanh Cong). Determine the greatest constant k such that the following inequality holds for all non-negative real numbers a, b, c:

$$\frac{a^2+b^2+c^2}{ab+bc+ca}+k\frac{a^2b+b^2c+c^2a}{ab^2+bc^2+ca^2}\geq 1+k$$

SOLUTION.

Assume that $p=1$. After changing, we need to prove that:

$$\begin{aligned}
(1-3q)^2(q-3r)^2 &\geq \left[1-(2k+3)q\right]^2(q^2+8qr-27r^2-4q^3-3p^3r\\
\Leftrightarrow f(r)=4(Ar^2+Br+C) &\geq 0
\end{aligned}$$

For:

$$\begin{aligned}A &= 27q^2k^2+81q^2k+81q^2-27qk-54q+9\\ B &= -18q^3k^2-54q^3k-54q^3+4q^2k^2+30q^2k+45q^2-4qk-12q+1\\ C &= q^3(4q^2k^2+12q^2k+9q^2-qk^2-7qk-6q+k+1)\end{aligned}$$

Observe that:

$$\Delta = -(3q-1)^2(3q-1+2qk)^2(12q^3k^2+36q^3k+36q^3-4q^2k^2-24q^2k-36q^2+4qk+12q-1)$$

Thus we can find that $k_{max}=k_0=\dfrac{9\sqrt[3]{3}}{8}+\dfrac{3\sqrt[3]{9}}{8}+\dfrac{3}{8}\approx 2.777562200$
+) There are some similar inequalities:

$$\begin{aligned}\frac{a^2+b^2+c^2}{ab+bc+ca}+k\frac{a^3b+b^3c+c^3a}{ab^3+bc^3+ca^3} &\geq 1+k\\ \frac{a^2+b^2+c^2}{ab+bc+ca}+k\frac{a^4b+b^4c+c^4a}{ab^4+bc^4+ca^4} &\geq 1+k\\ \frac{a^2+b^2+c^2}{ab+bc+ca}+k\frac{a^3b^2+b^3c^2+c^3a^2}{a^2b^3+b^2c^3+c^2a^3} &\geq 1+k\end{aligned}$$

<u>Example 13:</u> (Bach Ngoc Thanh Cong). Let a,b,c be non-negative real numbers, find the greatest constant k such that the following inequality holds:

$$\frac{3(a^2+b^2+c^2)}{(a+b+c)^2}+k\frac{a^4b+b^4c+c^4a}{a^3b^2+b^3c^2+c^3a^2}\geq 1+k$$

SOLUTION. The inequality is equivalent to:

$$k\sum_{sym}a^4(b+c)+\left[\frac{3\sum\limits_{sym}a^2}{\left(\sum\limits_{sym}\right)^2}-1-k\right]\sum_{sym}a^3(b^2+c^2)\geq\left[\left(\frac{3\sum\limits_{sym}a^2}{\left(\sum\limits_{sym}\right)^2}-1-k\right)\sum_{sym}bc-k\Big(\sum_{sym}a^2+\sum_{sym}bc\Big)\right]\prod_{cyc}(a-b)$$

We only need to consider the case $RHS\geq 0$. Without loss of generality, assume that $p=1$, then rewrite the inequality as:

$$\begin{aligned}\left[k(q-3q^2+5qr-r)+(2-6q-k)(q^2-qr-2r)\right]^2 &\geq (2q-6q^2-k)^2(q^2+18qr-27r^2-4q^3-4r)\\ \Leftrightarrow f(r)=4(Ar^2+Br+C) &\geq 0\end{aligned}$$

For:

$$\begin{aligned}A &= 252q^4+18kq^3-132q^3+9k^2q^2+114q^2k+40q^2+3qk^2-34qk-20q+7k^2-2k+4\\ B &= -180q^5-30kq^4+120q^4-12k^2q^3-68kq^3-20q^3+k^2q^2+44q^2k-4qk^2-6qk+k^2\\ C &= q^3(36q^4-24q^3+24q^2k+4q^2+4qk^2-14qk-k^2+2k)\end{aligned}$$

We have:

$$\Delta = -(3q-1)^2(k-2q+6q^2)^2(112q^5+8kq^4-84q^4+4k^2q^3+60kq^3+16q^3+3k^2q^2-44q^2k+2qk^2+8qk-k^2)$$

Hence we can find that $k_{max} = k_0 \approx 0.89985123$

+) There are some inequalities with the similar form:

$$\begin{aligned}
\frac{a^2+b^2+c^2}{ab+bc+ca} + k\frac{a^4b+b^4c+c^4a}{a^2b^3+b^2c^3+c^2a^3} &\geq 1+k \\
\frac{a^2+b^2+c^2}{ab+bc+ca} + k\frac{a^2b^3+b^2c^3+c^2a^3}{a^4b+b^4c+c^4a} &\geq 1+k \\
\frac{a^2+b^2+c^2}{ab+bc+ca} + k\frac{a^3b^2+b^3c^2+c^3a^2}{a^4b+b^4c+c^4a} &\geq 1+k
\end{aligned}$$

$\bigtriangledown\bigtriangledown$ Notice that in four previous examples, you can replace $\frac{a^2+b^2+c^2}{ab+bc+ca}$ by $\frac{3(a^2+b^2+c^2)}{(a+b+c)^2}$ or vice versa, and you can do the new inequality similarly.

Example 14: (Bach Ngoc Thanh Cong). Let $a, b, c \geq 0$, determine the greatest constant k such that the following inequality holds:

$$\frac{a^3}{b} + \frac{b^3}{c} + \frac{c^3}{a} + k(ab+bc+ca) \geq (k+1)(a^2+b^2+c^2)$$

SOLUTION. Suppose that $p = 1$, similarly to the previous examples, we have:

$$f(r) = 4(Ar^2 + Br + C) \geq 0$$

In which:

$$\begin{aligned}
A &= 9q^2k^2 + 27q^2k + 27q^2 - 6qk^2 - 27q - 18qk + k^2 + 3k + 9 \\
B &= -9q^3k - 18q^3 + 6q^2k + 19q^2 - qk - 8q + 1 \\
C &= q^5
\end{aligned}$$

Observe that:

$$\Delta = -(3q-1)^2(q-1)^2(4q^3k^2 + 12q^3k + 12q^3 - q^2k^2 - 12q^2k - 16q^2 + 2qk + 8q - 1)$$

Thus we can find that $k_{max} = k_0 = \dfrac{\sqrt[3]{1828+372\sqrt{93}}}{6} - \dfrac{106}{3\sqrt[3]{1828+372\sqrt{93}}} + \dfrac{5}{3} \approx 2.581412182$

Example 15: (Bach Ngoc Thanh Cong). Determine the greatest constant k such that the following inequality holds for any non-negative real numbers a, b, c:

$$3k\frac{a^4b+b^4c+c^4a}{ab^2+bc^2+ca^2} + a^2 + b^2 + c^2 \geq (k+1)(ab+bc+ca)$$

SOLUTION.

Suppose that $p = 1$, similarly, we have:

$$f(r) = 4(Ar^2 + Br + C)$$

In which:

$$\begin{aligned} A &= 108k^2q^2 + 81q^2 - 108k^2q + 108kq - 54q + 63k^2 - 36k + 9 \\ B &= -108k^2q^3 - 18kq^3 - 54q^3 + 100k^2q^2 - 69kq^2 + 45q^2 + -57k^2q - 43kq + 112q + 9k^2 - 6k \\ C &= q^2(4k^2q^3 - 12kq^3 + 9q^3 + 12k^2q^2 + 40kq^2 - 6q^2 - 3k^2q - 21kq + q + 3k) \end{aligned}$$

We have:

$$\Delta = -(3q-1)^2)(-3q+1+2kq-3k)^2(48k^2q^3+36q^3-52k^2q^2+48kq^2-36q^2+48k^2q-28kq+12q-9k^2+6k-1)$$

Hence $k_{max} = k_0 \approx 7.698078389$

+) Similar inequality:

$$3k\frac{a^4b + b^4c + c^4a}{a^2b + b^2c + c^2a} + a^2 + b^2 + c^2 \geq (k+1)(ab + bc + ca)$$

Example 16: (Bach Ngoc Thanh Cong). Find the greatest constant k such that the following inequality holds for all $a, b, c \geq 0$:

$$3k\frac{a^3b + b^3c + c^3a}{ab + bc + ca} + a^2 + b^2 + c^2 \geq (k+1)(ab + bc + ca)$$

SOLUTION.

Suppose that $p = 1$, then similarly, we have:

$$f(r) = 4(Ar^2 + Br + C)$$

For:

$$\begin{aligned} A &= 63k^2 \\ B &= 12k^2q^2 + 9kq^2 - 45qk^2 - 3kq + 9k^2 \\ C &= q^2(16k^2q^2 + 24kq^2 + 9q^2 - 3qk^2 - 17kq - 6q + 3k + 1) \end{aligned}$$

Observe that:

$$\Delta = -27k^2(3q-1)^2(16k^2q^2 + 24kq^2 + 9q^2 + 12qk^2 + 2kq - 3k^2)$$

Thus $k_{min} = k_0 \approx -0.3079785278$

+) Similar inequality:

$$3k\frac{a^2b+b^2c+c^2a}{a+b+c} + a^2+b^2+c^2 \geq (k+1)(ab+bc+ca)$$

<u>*Example 17:*</u> (Vo Quoc Ba Can). Prove the following inequality for $a, b, c \geq 0$:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq 3\left(\frac{a^2+b^2+c^2}{ab+bc+ca}\right)^k \quad \text{for} \quad k = \frac{2}{3}$$

SOLUTION.

This inequality is not so hard, you can do it by this method yourself, now we consider some general problems.

<u>General problem 1:</u> Find the greatest constant k such that the following inequality holds for any non-negative real numbers a, b, c:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+k \geq (3+k)\left(\frac{a^2+b^2+c^2}{ab+bc+ca}\right)^{\frac{2}{3}}$$

Solution.

Assume that $q = 1$, then let $t = \sqrt[3]{p^2-2}$, similarly, we can rewrite the inequality as:

$$f(r) = 4(Ar^2+Br+C) \geq 0$$

In which:

$$\begin{aligned} A &= (k+3)^2t^4 - (2k^2+3k-9)t^2 + (k^2-3k+9) \\ B &= \sqrt{t^3+2}\Big[t^3-(k+3)t^2+k-4\Big] \\ C &= 1 \end{aligned}$$

We have:

$$\begin{aligned} \Delta &= (t-1)^2(t^7-2t^6k-4t^6+t^5k^2+2t^5k+2t^4k^2+8t^4k-2t^4+t^3k^2+12t^3k+8t^3-2t^2k^2+4t^2k \\ &\quad -4tk^2-8tk-8t-2k^2-4k-4) \end{aligned}$$

Hence we can find that $k_{max} = k_0 \approx 0.3820494092$, the general problem 1 is solved.

<u>General problem 2:</u> Let $a, b, c > 0$, determine the greatest constant k such that the following inequality holds:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq 3\left(\frac{a^2+b^2+c^2}{ab+bc+ca}\right)^k$$

Solution.
Similarly, assume that $q = 1$, then let $t = p^2 - 2$, we have:

$$f(r) = 4(Ar^2 + Br + C)$$

For:

$$\begin{aligned} A &= 9(t^{2k} + t^k + 1) \\ B &= \sqrt{t+2}\Big(t - 3t^k - 4) \\ C &= 1 \end{aligned}$$

General Problem 3: Let a, b, c be positive real numbers, find the necessary and sufficient relation between k and t such that the following inequality holds:

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} + k \geq (3+k)\left(\frac{a^2+b^2+c^2}{ab+bc+ca}\right)^t$$

Example 17: (Bach Ngoc Thanh Cong). Determine the greatest constant k such that the following inequality holds for any positive real numbers a, b, c:

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} + k(a+b+c) \geq (3+3k)\sqrt{\frac{a^2+b^2+c^2}{3}}$$

SOLUTION.
Similarly, assume that $p = 1$, then let $t = 3 - 6q$, rewrite the inequality as:

$$f(r) = Ar^2 + Br + C \geq 0$$

In which:

$$\begin{aligned} A &= 1296(t^2k^2 + 2t^2k + t^2 - 2k^2t - tk + t + k^2 - k + 7) \\ B &= 4(18t^5k + 18t^5 - 18t^4k + 9t^4 - 54t^3k - 54t^3 + 54t^2k + 216t^2 - 405) \\ C &= t^8 - 12t^6 + 54t^4 - 108t^2 + 81 \end{aligned}$$

Observe that:

$$\begin{aligned} \Delta' &= 972(t-1)^2(8t^6k^2 + 16t^6k - t^6 + 60t^5k + 42t^5 - 60t^4k^2 - 60t^4k + 153t^4 - 288t^3k + 36t^3 \\ &\quad +144t^2k^2 - 279t^2 + 324tk - 270t - 108k^2 + 108k - 81) \end{aligned}$$

Hence we find that $k_{max} = k_0 \approx 4.356328100$

+) General problem:

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} + k(a+b+c) \geq (3+3k)\left(\frac{a^2+b^2+c^2}{3}\right)^t$$

3 Propose problems:

There are some problems proved by this method:

Problem 1: (Pham Sinh Tan). Find the greatest constant k such that the following inequalty holds for all $a, b, c \geq$:

$$\frac{(k+1)^3}{8}(a^2+b^2+c^2)^2 \geq a(a+kb)^3 + b(b+kc)^3 + c(c+ka)^3$$

Problem 2: (Vo Quoc Ba Can). Let a, b, c be three potitive real numbers. Prove that:

$$\begin{aligned} \frac{8}{3}(a^2+b^2+c^2)^2 - \frac{1}{3}\Big(a^4+b^4+c^4-abc(a+b+c)\Big) &\geq a(a+b^3+b(b+c)^3+c(c+a)^3 \\ &\geq \frac{8}{27}(a+b+c)^4 + \frac{1}{125}\Big(a^4+b^4+c^4-abc(a+b+c)\Big) \end{aligned}$$

Problem 3: (Bach Ngoc Thanh Cong). Determine the greatest constant k such that the following inequality holds for any non-negative real numbers a, b, c:

$$\sqrt{\frac{a^2+b^2+c^2}{ab+bc+ca}} + k\frac{3abc}{ab^2+bc^2+ca^2} \geq 1+k$$

+) Note that this is a different form of the example 10, then you can realize that in the examples 10,11,12,13 you can replace $\dfrac{a^2+b^2+c^2}{ab+bc+ca}$ by $\left(\dfrac{a^2+b^2+c^2}{ab+bc+ca}\right)^t$ $\Bigg($ and of course, similarly to $\dfrac{3(a^2+b^2+c^2)}{(a+b+c)^2}\Bigg)$, and you can do the new inequality your self, some are not so hard.

Problem 4:Let a, b, c be positive real numbers, determine the necessary and sufficient relation between k and t such that the following inequality holds:

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} + k \geq (3+k)\left(\frac{3(a^2+b^2+c^2)}{(a+b+c)^2}\right)^t$$