99.09 A family of Napoleon triangles associated with the Kiepert configuration

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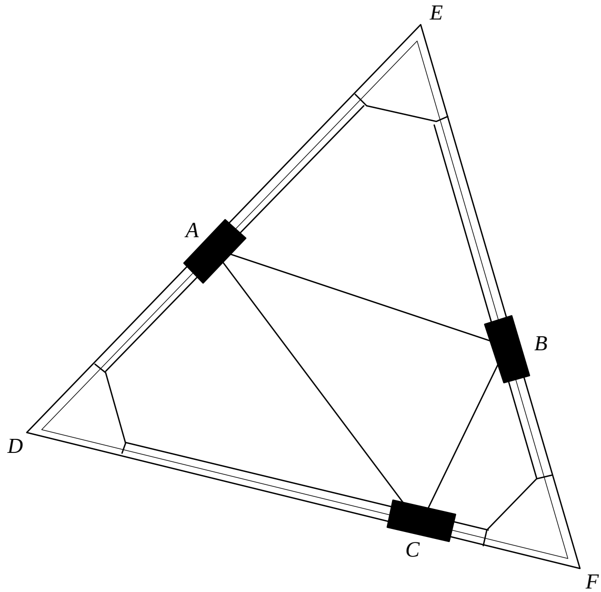

FIGURE 2: A physical device to find the maximal outscribed equilateral triangle.
At A, B and C, frictionless sleeves are free to pivot.
Through these sleeves pass telescoping spring-loaded bars.
The bars are welded rigidly to 60° formers at D, E and F.

References

1. Mark Levi, *The mathematical mechanic*, Princeton University Press (2009).
2. Fengming Dong, Dongsheng Zhao and Weng Kin Ho, On the largest outscribed equilateral triangle, *Math. Gaz*. **98** (March 2014) pp. 79-84.

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99.09 A family of Napoleon triangles associated with the Kiepert configuration

A classic theorem in plane Euclidean geometry, often doubtfully attributed to Napoleon, states that the centres of equilateral triangles erected outwardly on the sides of a triangle ΔABC form an equilateral triangle. We shall denote these centres by N_A, N_B and N_C. This is easily proved from the fact that the line of centres of two intersecting circles is the perpendicular bisector of their common chord. It is also true that the new vertices of these triangles form a triangle which is in perspective with ΔABC. These results also hold when the initial triangles are erected facing inwards. The common centre of these triangles is the centroid G of ΔABC.

A generalisation of this result is due to Ludwig Kiepert. This uses similar isosceles triangles rather than equilateral triangles, and the new vertices again form a triangle in perspective with ΔABC. As the base angles vary (being allowed to become negative to give the inward-facing case), the locus of the perspector is the Kiepert hyperbola. Details of all these results can be found in [1].

In what follows, we produce some more equilateral triangles from this configuration. Let the base angles of the isosceles triangles be α and let $t = \tan\alpha$. Denote the new vertices formed by the isosceles triangles be A_0, B_0 and C_0 and define the point A^* on the ray AA_0 by setting $AA^* = \dfrac{2}{3 - \sqrt{3}t}AA_0$, where a negative sign indicates that A^* is on the opposite side of A to A_0. Points B^* and C^* are defined analogously. In the diagram which follows, α is taken as smaller than 30°, but this does not affect the argument. Denote the midpoint of BC by A_1.

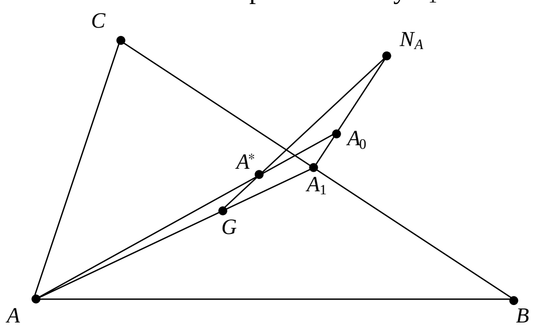

FIGURE 1

Then a series of simple calculations shows that

$$\frac{A_0A^*}{A^*A} = \frac{1 - \sqrt{3}t}{2}, \quad \frac{A_1N_A}{N_AA_0} = \frac{1}{\sqrt{3}t - 1} \text{ and, of course, } \frac{AG}{GA_1} = 2$$

so by Menelaus's theorem on ΔAA_1A_0 it follows that G, A^*, N_A are collinear.

Exactly the same argument works for G, B^*, N_B and G, C^*, N_C and another application of Menelaus shows that the ratios $GA^* : GN_A$ are independent of which vertex is being used and therefore equal. It therefore follows that, since G is the circumcentre of the Napoleon triangle, $\Delta A^*B^*C^*$ is homothetic to $\Delta N_AN_BN_C$ and is therefore equilateral with centre G. In the special case when $\alpha = 30°$ the triangles $\Delta A^*B^*C^*$ and $\Delta N_AN_BN_C$ coincide. If $\alpha = 60°$, AA_0 is parallel to A_1N_A and A^* is at infinity.

If the same method is used with $AA^* = \dfrac{2}{3 + \sqrt{3}t}AA_0$, we obtain a similar configuration with $\Delta A^*B^*C^*$ homothetic to $\Delta N_aN_bN_c$, the inner Napoleon triangle, formed by the centres of inwardly-facing equilateral triangles.

Reference

1. G. Leversha, *The geometry of the triangle*, UKMT (2013).

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99.10 Proof without words: $T_1 + T_2 + \ldots + T_n = \binom{n+2}{3}$

Underneath the symbols $T_1, T_2, \ldots, T_n$ place $(n + 2)$ balls, with two balls to the left of T_1. Three of these balls can be selected in $\binom{n+2}{3}$ ways. In the example shown, the second, fifth and $(m + 2)$th balls have been selected.

We interpret the third of these as the triangular number T_m and then the first two balls represent two numbers in the range from 1 to $m + 1$. We shall interpret these two numbers as an instruction for defining a unique dot amongst the T_m dots in the triangular array. The diagram below shows how this is done in the case when $m = 10$, but it is clearly completely general.

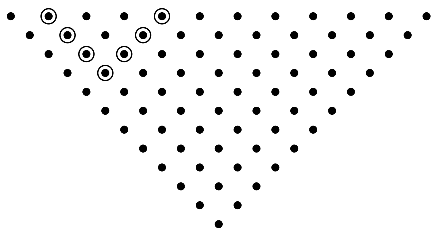

FIGURE 1

The diagram shows the array representing T_{m+1}, which can be thought of as a top row of $m + 1$ dots below which is the array for T_m. The second and fifth dots in the top row have been highlighted since the second and fifth balls were selected. Now these are used to pinpoint the second dot in the fourth row of T_{11} as the intersection of two diagonals in different directions. This is also the second dot in the third row of T_{10}. We now have a one-to-one correspondence between the $\binom{n+2}{3}$ choices of three objects from $n + 2$ objects and the totality of points in all the triangular arrays from T_1 to T_n, and this is enough to prove the result.