Problem 11805
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Proposed by G. Glebov and S. Fraser (Canada).

(a) Show that

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(3k+1)^3}+\sum_{k=0}^{\infty}\frac{(-1)^k}{(3k+2)^3}=\frac{5\pi^3\sqrt{3}}{243},$$
$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(3k+1)^3}-\sum_{k=0}^{\infty}\frac{(-1)^k}{(3k+2)^3}=\frac{13}{18}\zeta(3).$$

(b) Prove that

$$\zeta(3)=\frac{9}{13}\int_0^1\frac{(\ln x)^2}{x^3+1}\,dx-\frac{18}{13}\sum_{k=0}^{\infty}\frac{(-1)^k}{(3k+2)^3}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

(a) We note that

$$A:=\sum_{k=0}^{\infty}\frac{(-1)^k}{(3k+1)^3}=\sum_{k=0}^{\infty}\frac{1}{(6k+1)^3}-\sum_{k=0}^{\infty}\frac{1}{(6k+4)^3},$$
$$B:=\sum_{k=0}^{\infty}\frac{(-1)^k}{(3k+2)^3}=\sum_{k=0}^{\infty}\frac{1}{(6k+2)^3}-\sum_{k=0}^{\infty}\frac{1}{(6k+5)^3}.$$

Hence

$$A-B=-\mathrm{Li}_3(-1)+\sum_{k=1}^{\infty}\frac{1}{6k^3}-\sum_{k=0}^{\infty}\frac{1}{(6k+3)^3}=-\frac{26}{27}\mathrm{Li}_3(-1)=\frac{13}{18}\zeta(3).$$

Moreover, by letting $\omega=e^{i\pi/3}$ we have

$$\begin{aligned}\mathrm{Li}_3(\omega)=\sum_{k=1}^{\infty}\frac{\omega^k}{k^3}&=\sum_{k=1}^{\infty}\frac{1}{6k^3}-\sum_{k=0}^{\infty}\frac{1}{(6k+3)^3}+\omega\sum_{k=0}^{\infty}\frac{1}{(6k+1)^3}-\omega\sum_{k=0}^{\infty}\frac{1}{(6k+4)^3}\\&\quad-\overline{\omega}\sum_{k=0}^{\infty}\frac{1}{(6k+2)^3}+\overline{\omega}\sum_{k=0}^{\infty}\frac{1}{(6k+5)^3}\\&=-\frac{\zeta(3)}{36}+\omega A-\overline{\omega}B=-\frac{\zeta(3)}{36}+\frac{A-B}{2}+\frac{i\sqrt{3}(A+B)}{2}.\end{aligned}$$

It is known that polylogarithms satisfy the following identity

$$\mathrm{Li}_n(e^{2\pi ix})+(-1)^n\mathrm{Li}_n(e^{-2\pi ix})=-\frac{(2\pi i)^n}{n!}B_n(x)$$

where B_n is the Bernoulli polynomial of degree $n\geq 1$. For $n=3$ and $x=1/6$, it becomes

$$2i\mathrm{Im}\left(\mathrm{Li}_3(\omega)\right)=\mathrm{Li}_3(\omega)-\mathrm{Li}_3(\overline{\omega})=\frac{4\pi^3 i}{3}B_3(1/6)=\frac{5\pi^3 i}{81}.$$

Hence

$$A+B=\frac{2\sqrt{3}}{3}\,\mathrm{Im}(\mathrm{Li}_3(\omega))=\frac{5\pi^3\sqrt{3}}{243}.$$

(b) If $|z| = 1$ and $z \neq 1$ then

$$\begin{aligned}\int_0^1 \frac{(\ln x)^2}{x-z}\,dx &= -\frac{1}{z}\int_0^1 \frac{(\ln x)^2}{1-(x/z)}\,dx = -\frac{1}{z}\int_0^1 (\ln x)^2 \sum_{k=0}^{\infty} \frac{x^k}{z^k}\,dx = -\sum_{k=0}^{\infty} \frac{1}{z^{k+1}} \int_0^1 (\ln x)^2 x^k\,dx \\ &= -\sum_{k=0}^{\infty} \frac{1}{z^{k+1}} \left[x^{k+1} \left(\frac{(\ln x)^2}{k+1} - \frac{2\ln x}{(k+1)^2} + \frac{2}{(k+1)^3} \right) \right]_0^1 \\ &= -2\sum_{k=0}^{\infty} \frac{1}{z^{k+1}(k+1)^3} = -2\mathrm{Li}_3(\overline{z}).\end{aligned}$$

Hence

$$\begin{aligned}\int_0^1 \frac{(\ln x)^2}{x^3+1}\,dx &= \frac{1}{3}\int_0^1 \frac{(\ln x)^2}{x+1}\,dx - \frac{\omega}{3}\int_0^1 \frac{(\ln x)^2}{x-\omega}\,dx - \frac{\overline{\omega}}{3}\int_0^1 \frac{(\ln x)^2}{x-\overline{\omega}}\,dx \\ &= \frac{2}{3}\left(-\mathrm{Li}_3(-1) + \omega \mathrm{Li}_3(\overline{\omega}) + \overline{\omega}\mathrm{Li}_3(\omega)\right) \\ &= \frac{\zeta(3)}{2} + \frac{2}{3}\,\mathrm{Re}\left(\mathrm{Li}_3(\omega)\right) + \frac{2\sqrt{3}}{3}\,\mathrm{Im}\left(\mathrm{Li}_3(\omega)\right) \\ &= \frac{\zeta(3)}{2} + \frac{2}{3}\left(-\frac{\zeta(3)}{36} + \frac{A-B}{2}\right) + (A+B) \\ &= \frac{13\zeta(3)}{18} + \frac{5\sqrt{3}\pi^3}{243}\end{aligned}$$

which is equivalent to the required equality. $\square$