Problem 11798. [AMM, October 2014, p.738]. *Proposed by Finbarr Holland, University College Cork, Cork, Ireland.* For positive integers n, let f_n be the polynomial given by

$$f_n(x) = \sum_{r=0}^{n} \binom{n}{r} x^{\lfloor r/2 \rfloor} .$$

(a) Prove that if $(n+1)$ is prime, then f_n is irreducible over $\mathbb{Q}$.

(b) Prove that for all n (whether $n+1$ is prime or not),

$$f_n(1+x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} 2^{n-2k} x^k .$$

Solution by Borislav Karaivanov, Lexington, SC, and Tzvetalin S. Vassilev, Nipissing University, North Bay, Ontario, Canada. Considering the fact that for even r we have $\lfloor r/2 \rfloor = \lfloor (r+1)/2 \rfloor = r/2$, we rewrite the definition of f_n as follows

$$f_n(x) = \sum_{r=0}^{n} \binom{n}{r} x^{\lfloor r/2 \rfloor} = \sum_{r=0}^{\lfloor n/2 \rfloor} \left[\binom{n}{2r} + \binom{n}{2r+1} \right] x^r = \sum_{r=0}^{\lfloor n/2 \rfloor} \binom{n+1}{2r+1} x^r.$$

Now, if $n+1$ is an odd prime, say p, the claim in part (a) follows by the Eisenstein's criterion. Indeed, the leading coefficient is 1, every other coefficient being of the form $\binom{p}{i}$, $1 \le i < p$, is divisible by p, and the constant term $\binom{p}{1} = p$ is not divisible by p^2. In the case $n+1 = 2$, the respective polynomial, $f_1(x) = 2$ is clearly irreducible over the rationals, as well.

To address part (b), we rewrite

$$f_n(1+x) = \sum_{r=0}^{\lfloor n/2 \rfloor} \binom{n+1}{2r+1} (1+x)^r = \sum_{r=0}^{\lfloor n/2 \rfloor} \binom{n+1}{2r+1} \sum_{k=0}^{r} \binom{r}{k} x^k .$$

Reordering the terms by the powers of the variable, we obtain

$$f_n(1+x) = \sum_{i=0}^{\lfloor n/2 \rfloor} x^i \cdot \sum_{j=i}^{\lfloor n/2 \rfloor} \binom{j}{i} \binom{n+1}{2j+1}.$$

So, we only need to prove that

$$\sum_{j=i}^{\lfloor n/2 \rfloor} \binom{j}{i} \binom{n+1}{2j+1} = \binom{n-i}{i} \cdot 2^{n-2i}$$

which we show by combinatorial means. Indeed, both sides count the $(n+1)$-sequences of white, green, and red balls with exactly i red balls which separate the green balls into $i+1$ groups of odd lengths. Starting with $n+1$ white balls, the left-hand side counts the number of ways of painting an odd number, say $2j+1$, of at least $2i+1$ balls in green, and further repainting in red i of the j even-indexed green balls. To count the same sequences, the right-hand side sets aside $i+1$ balls, paints i of the remaining $n-i$ balls in red, and paints a subset of the now remaining $n-2i$ balls in green. Next, the $i+1$ white balls set aside are inserted, one immediately before every red ball and one at the end of the sequence, and some of them are painted green if needed to ensure that the green balls form groups of odd lengths. $\square$