Problem 11797. [AMM, October 2014, p.738]. *Proposed by Zhang Yun, Xi'an, Shaanxi Province, China.* Let A_1, A_2, A_3, and A_4 be the vertices of a tetrahedron. Let h_k be the length of the altitude from A_k to the plane of the opposite face, and let r be the radius of the inscribed sphere. Prove that

$$\sum_{k=1}^{4} \frac{h_k - r}{h_k + r} \geq \frac{12}{5} .$$

Solution by Borislav Karaivanov, Lexington, SC, and Tzvetalin S. Vassilev, Nipissing University, North Bay, Ontario, Canada.

Let O be the centre of the inscribed sphere, and V_k denote the volume of the tetrahedron formed by O and the vertices of the original tetrahedron except A_k. Let V be the volume of the tetrahedron $A_1A_2A_3A_4$. We have

$$\frac{h_k - r}{h_k + r} = \frac{1 - \frac{r}{h_k}}{1 + \frac{r}{h_k}} = \frac{1 - \frac{V_k}{V}}{1 + \frac{V_k}{V}}$$

Having in mind that

$$V_1 + V_2 + V_3 + V_4 = V \Leftrightarrow \frac{V_1}{V} + \frac{V_2}{V} + \frac{V_3}{V} + \frac{V_4}{V} = 1 ,$$

The problem reduces to showing that for positive numbers $\alpha, \beta, \gamma, \delta$ such that $\alpha + \beta + \gamma + \delta = 1$, we have

$$\frac{1-\alpha}{1+\alpha} + \frac{1-\beta}{1+\beta} + \frac{1-\gamma}{1+\gamma} + \frac{1-\delta}{1+\delta} \geq \frac{12}{5}$$

This follows from the convexity of the function $f(x) = \dfrac{1-x}{1+x}$ on the interval $(0, 1)$:

$$\frac{f(\alpha) + f(\beta) + f(\gamma) + f(\delta)}{4} \geq f\left(\frac{\alpha + \beta + \gamma + \delta}{4}\right) \Leftrightarrow$$

$$\frac{1-\alpha}{1+\alpha} + \frac{1-\beta}{1+\beta} + \frac{1-\gamma}{1+\gamma} + \frac{1-\delta}{1+\delta} \geq 4 \cdot f\left(\frac{1}{4}\right) = 4 \cdot \frac{3}{5} = \frac{12}{5}$$

Equality is attained if and only if $\alpha = \beta = \gamma = \delta = \frac{1}{4}$ which is equivalent to $V_1 = V_2 = V_3 = V_4 = V/4$, or in our terms, when the faces of the tetrahedron have equal area.

Please note that this problem is identical to problem number 11783, published in June-July 2014 edition of the Monthly on page 549.