Problem 11796. [AMM, October 2014]. *Proposed by G. Glebov, Burnaby, Canada.* Find

$$\int_0^\infty \frac{\sin((2n+1)x)}{\sin x} e^{-\alpha x} x^{m-1} dx$$

in terms of α, m, and n, when $\alpha > 0$, $m > 0$, and n is a nonnegative integer.

Solution by Borislav Karaivanov, Lexington, SC, and Tzvetalin S. Vassilev, Nipissing University, North Bay, Ontario, Canada. Using the well-known Dirichlet kernel representation $\frac{\sin(2n+1)x}{\sin x} = \sum_{k=-n}^{n} e^{2kxi}$ and the formula $\Gamma(m) = z^m \int_0^\infty t^{m-1} e^{-zt} dt$ valid when Re(m) > 0 and Re(z) > 0, (see 6.1.1 in [1]), we find

$$\int_0^\infty \frac{\sin(2n+1)x}{\sin x} e^{-\alpha x} x^{m-1} dx = \sum_{k=-n}^{n} \int_0^\infty e^{-(\alpha-2ki)x} x^{m-1} dx = \Gamma(m) \sum_{k=-n}^{n} \frac{1}{(\alpha-2ki)^m}. \quad \square$$

References

[1] M. Abramowitz, I. Stegun, eds., *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, 1972, New York: Dover.