Problem 11791. [AMM, August-September 2014]. *Proposed by M. Stofka, Bratislava, Slovakia.* Show that for $n \geq 1$,

$$\sum_{k=1}^{n} \binom{6n+1}{6k-2} B_{6k-2} = -\frac{6n+1}{6},$$

where B_n denotes the n-th Bernoulli number.

Solution by Borislav Karaivanov, Lexington, SC. Essentially, this is a known result. In Wikipedia (*http://en.wikipedia.org/wiki/Bernoulli_number*) it is listed under Ramanujan's congruences as

$$\binom{m+3}{m} B_m = -\frac{m+3}{6} - \sum_{j=1}^{(m-4)/6} \binom{m+3}{m-6j} B_{m-6j}, \qquad \text{if } m \equiv 4 \pmod 6$$

(set $m = 6n-2$ and $j = n-k$). It also follows from [1],(4.5), (after correcting a typo):

$$\binom{6k+m+3}{3} B_{6k+m} = -\sum_{j=0}^{k-1} \binom{6k+m+3}{6j+m} B_{6j+m} + \frac{1}{6}(6k+m+3)w_{m-1},$$

where $w_m = 1 + (-1)^m(\theta^m + \theta^{6-m})$ with $\theta = e^{-\frac{2\pi i}{3}}$ (set $m = 4$, $k = n-1$, and $j = k-1$). As indicated in [1], in some form, it is also in [2], pp. 3-4, and [3], pp. 136-137. $\square$

References

[1] F. T. Howard. "Applications of a Recurrence for the Bernoulli Numbers." J. Number Theory 52 (1995):157-172.

[2] S. Ramanujan. "Some Properties of Bernoulli's Numbers." Journal of the Indian Mathematical Society 3 (1911):219-234.

[3] J. Riordan. Combinatorial Identities. New York: Wiley, 1968.