Problem 11788. [AMM, June-July 2014]. *Proposed by S. Andriopoulos, Eleia, Greece.* Let n be a positive integer, and suppose that $0 < y_i \le x_i < 1$ for $1 \le i \le n$. Prove that

$$\frac{\log x_1 + \cdots + \log x_n}{\log y_1 + \cdots + \log y_n} \le \sqrt{\frac{1-x_1}{1-y_1} + \cdots + \frac{1-x_n}{1-y_n}}.$$

Solution by Borislav Karaivanov, Lexington, SC. For $n = 1$ the inequality is equivalent to $\frac{\log x_1}{\sqrt{1-x_1}} \ge \frac{\log y_1}{\sqrt{1-y_1}}$. We consider $f(x) = \frac{\log x}{\sqrt{1-x}}$ on $(0, 1)$. Its derivative $f'(x) = \frac{x \log x + 2 - 2x}{2x\sqrt{1-x}^3}$ has positive denominator. The numerator $g(x) = x \log x + 2 - 2x$ is positive too. Indeed, $g'(x) = \log x - 1 < 0$ which means that g strictly decreasing and, hence, $g(x) > g(1) = 0$ on $(0, 1)$. Thus f is strictly increasing on $(0, 1)$ which yields the desired inequality with equality attained if and only if $x_1 = y_1$.

For $n \ge 2$ we rewrite and apply the case $n = 1$ to get

$$\frac{\log x_1 + \cdots + \log x_n}{\log y_1 + \cdots + \log y_n} = \frac{\log x_1 \dots x_n}{\log y_1 \dots y_n} \le \sqrt{\frac{1 - x_1 \dots x_n}{1 - y_1 \dots y_n}} < \sqrt{\frac{1-x_1}{1-y_1} + \cdots + \frac{1-x_n}{1-y_n}},$$

where the last inequality is easily obtained by induction from the simple inequality derived as follows

$$\frac{1 - x_1x_2}{1 - y_1y_2} < \frac{(1-x_1) + (1-x_2)}{1 - y_1y_2} = \frac{1-x_1}{1-y_1y_2} + \frac{1-x_2}{1-y_1y_2} < \frac{1-x_1}{1-y_1} + \frac{1-x_2}{1-y_2}$$

for $x_1, x_2, y_1, y_2 \in (0, 1)$. Clearly, for $n \ge 2$ equality is never attained. $\square$