Problem 11782
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Proposed by Zhang Yun (China).

Given a tetrahedron, let r denote the radius of its inscribed sphere. For $1 \leq k \leq 4$, let h_k denote the distance from the kth vertex to the plane of the opposite face. Prove that

$$\sum_{k=1}^{4} \frac{h_k - r}{h_k + r} \geq \frac{12}{5}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

The volume of the tetrahedron is given by

$$\frac{h_k A_k}{3} = \frac{rS}{3}$$

where A_k is the area of the face of opposite to the kth vertex and $S = \sum_{k=1}^{4} A_k$ is the surface area of the tetrahedron. Hence $h_k = r/t_k$ with $t_k = A_k/S \in (0, 1)$ and

$$\sum_{k=1}^{4} \frac{h_k - r}{h_k + r} = \sum_{k=1}^{4} f(t_k) \geq 4f\left(\frac{1}{4}\sum_{k=1}^{4} t_k\right) = 4f\left(\frac{1}{4}\right) = \frac{12}{5}$$

where $f(t) = (1-t)/(1+t)$ is a convex function in $[0, +\infty)$. $\square$