Problem 11782. [AMM, June-July 2014]. *Proposed by I. Gessel, Waltham, MA.* A signed binary representation of an integer m is a finite list $a_0, a_1, \ldots$ of elements of $\{-1, 0, 1\}$ such that $\sum a_i 2^i = m$. A signed binary representation is sparse if no two consecutive entries in the list are nonzero.
(a) Prove that every integer has a unique sparse representation.
(b) Prove that for all $m \in \mathbb{Z}$, every non-sparse signed binary representation of m has at least as many nonzero terms as the sparse representation.

Solution by Borislav Karaivanov, Lexington, SC. (a) *Existence.* For any $m \geq 0$, its binary representation is also a signed binary representation; for $m < 0$, negating each term in the binary representation of $|m|$ yields a signed binary representation of m. For the rest of the proof, we have *representation* stand for "signed binary representation" and *pair* for a "pair of consecutive nonzero terms". Next we show that every representation can be converted into a sparse one. For any representation we define d_1 to be the squared number of pairs and d_2 to be the sum $\sum(l-i)$ over all pairs (a_i, a_{i-1}), where l is the index of the left-most nonzero term. We define the "defect" d as $d_1 + d_2$ and use it to argue existence by induction. To this end we consider four kinds of sum-preserving transforms, each changing three consecutive terms in a representation, as indicated, while keeping the rest unchanged:

$$(0, 1, -1) \to (0, 0, 1) \tag{1}$$
$$(0, -1, 1) \to (0, 0, -1) \tag{2}$$
$$(0, 1, 1) \to (1, 0, -1) \tag{3}$$
$$(0, -1, -1) \to (-1, 0, 1). \tag{4}$$

If $d = 0$ the representation is sparse already. Suppose every representation with "defect" smaller than d can be made sparse with finite number of transform of the type (1)-(4). If the left-most pair matches (1) or (2), we get a representation with both d_1 and d_2 reduced because a pair disappears. Otherwise, either (3) or (4) applies to the left-most pair and we distinguish the following three situations:

(i) no nonzero terms exist to the left of the pair. In this case the left-most nonzero term moves to the left by 1 which increases each remaining summand in d_2 by 1, i.e., if p is the number of pairs, then d_2 is increased to $d_2 + p - 1$. In the same time, a pair is lost and d_1 is reduced from p^2 to $(p-1)^2$. Therefore, d changes from $p^2 + d_2$ to $p^2 + d_2 - p$, a reduction since p is at least 1.

(ii) there is a nonzero term to the left of the pair but the two terms immediately to the left are zeros. The pair is destroyed without creating a new one; both d_1 and d_2 get reduced.

(iii) immediately to the left of the pair we have $(1, 0)$ or $(-1, 0)$. The pair is destroyed but a new pair "closer" to the left-most nonzero term is created; d_1 remains the same but d_2 gets reduced.

Thus, in any case the "defect" gets reduced and the inductive hypothesis applies yielding the desired sparse representation.

Uniqueness. Suppose $\sum a_i 2^i$ and $\sum b_i 2^i$ are different sparse representations with $\sum a_i 2^i = \sum b_i 2^i$. Let n be the largest for which $a_n \neq b_n$. Without loss of generality,

$a_n > b_n$. Then

$$
\begin{aligned}
0 = \sum a_i 2^i - \sum b_i 2^i &\geq (a_n - b_n)2^n + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} (-1) \cdot 2^{n-2i} - \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} 1 \cdot 2^{n-2i} \\
&\geq 2^n - 2^{n+1} \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \left(\frac{1}{4}\right)^i = 2^n \left(1 - \frac{2}{3}\left(1 - 4^{-\lfloor \frac{n}{2} \rfloor}\right)\right) = \frac{1}{3}\left(2^n + 2^{n+1-2\lfloor \frac{n}{2} \rfloor}\right) \geq 1,
\end{aligned}
$$

where the sums in the lower bound include only every other term because the representations are sparse. This contradiction proves the uniqueness.

(b) This is immediate from the existence proof in part (a) since none of the four types of transforms (1)-(4) increases the number of nonzero terms. □