Problem 11775. [AMM, May 2014]. *Proposed by I. Sofair, Fredericksburg, VA.* Let $A_1, \ldots, A_k$ be finite sets. For $J \subseteq \{1, \ldots, k\}$, let $N_J = | \cup_{j \in J} A_j|$, and let $S_m = \sum_{J:|J|=m} N_J$.
(a) Express in terms of $S_1, \ldots, S_k$ the number of elements that belong to exactly m of the sets $A_1, \ldots, A_k$.
(b) Same question as in (a), except that we now require the number of elements belonging to at least m of the sets $A_1, \ldots, A_k$.

Solution by Borislav Karaivanov, Lexington, SC. (a) Let T_m be the number of elements that belong to exactly m of the sets $A_1, \ldots, A_k$. An element b counted by T_m is counted in S_i once for each i-tuple of sets drawn from $A_1, \ldots, A_k$ except for the i-tuples $(A_{j_1}, \ldots, A_{j_i})$ such that none of A_{j_s}'s contains b. Therefore, $S_i = \sum_{m=1}^{k} \left(\binom{k}{i} - \binom{k-m}{i}\right) T_m$. (Here and below, we use the convention that $\binom{n}{l} = 0$ when $n < l$, or $n < 0$, or $l < 0$.) Thus $S = (B - C)T$, where S and T are column vectors holding the S_i's and T_m's, and B and C are $k \times k$ matrices with entries given by $b_{im} = \binom{k}{i}$ and $c_{im} = \binom{k-m}{i}$, correspondingly. To solve the system for T, we left-multiply by the inverse D of $B - C$. The entries of D are given by $d_{mj} = (-1)^{k+1+m+j}\binom{j}{k-m}$. Indeed, for the entries of BD we have

$$\sum_{m=1}^{k} b_{im} d_{mj} = (-1)^{k+1+j} \binom{k}{i} \sum_{m=1}^{k} (-1)^m \binom{j}{k-m} = (-1)^{j+1} \binom{k}{i} \sum_{m=0}^{k-1} (-1)^m \binom{j}{m}$$

$$= (-1)^{j+1} \binom{k}{i} \left(\sum_{m=0}^{j} (-1)^m \binom{j}{m} - (-1)^j \binom{j}{j} \delta_{jk} \right) = \binom{k}{i} \delta_{jk},$$

where δ_{jk} is the Kronecker delta, and for the entries of CD we find

$$\sum_{m=1}^{k} c_{im} d_{mj} = (-1)^{k+1+j} \sum_{m=1}^{k} (-1)^m \binom{k-m}{i} \binom{j}{k-m}$$

$$= (-1)^{k+j+1} \binom{j}{i} \sum_{m=0}^{k-1} (-1)^m \binom{j-i}{m+j-k} = -\binom{j}{i} \sum_{m=j+1-k}^{j} (-1)^m \binom{j-i}{m}$$

$$= -\binom{j}{i} \left(\sum_{m=0}^{j-i} (-1)^m \binom{j-i}{m} - \binom{j-i}{0} \delta_{jk} \right) = -\binom{j}{i} (\delta_{ij} - \delta_{jk}).$$

Combining the last two results, we obtain $\sum_{m=1}^{k} (b_{im} - c_{im}) d_{mj} = \binom{k}{i}\delta_{jk} + \binom{j}{i}(\delta_{ij} - \delta_{jk}) = \left(\binom{k}{i} - \binom{j}{i}\right)\delta_{jk} + \binom{j}{i}\delta_{ij} = \delta_{ij}$. Therefore, $T = DS$, i.e., defining $S_0 = 0$, we have

$$T_m = \sum_{i=1}^{k} d_{mi} S_i = \sum_{i=1}^{k} (-1)^{k+1+m+i} \binom{i}{k-m} S_i = \sum_{i=k-m}^{k} (-1)^{k+1+m+i} \binom{i}{k-m} S_i.$$

(b) For the number U_m of elements belonging to at least m of the sets $A_1, \ldots, A_k$ we use part (a) to obtain

$$U_m = \sum_{j=m}^{k} T_j = \sum_{j=m}^{k} \sum_{i=1}^{k} (-1)^{k+1+j+i} \binom{i}{k-j} S_i = \sum_{i=1}^{k} \left((-1)^{i+1} S_i \sum_{j=0}^{k-m} (-1)^j \binom{i}{j} \right)$$

$$= \sum_{i=k-m+1}^{k} \left((-1)^{i+1} \sum_{j=0}^{k-m} (-1)^j \binom{i}{j} \right) S_i. \qquad \square$$