Asymmetry

V4-**1.** *Proposed by the editor.* Show that

$$\sum_{n\geq 1}\frac{(-1)^{n-1}}{n(n+1)\cdots(n+k)}=\frac{2^k}{k!}\left(\ln 2-\sum_{i=1}^{k}\frac{(1/2)^i}{i}\right)$$

where k is a nonnegative integer and for $k=0$ the second sum is considered to be 0.

Solution V4-1, by Omran Kouba, *Higher Institute for Applied Sciences and Technology, Damascus, Syria.*

The proposed problem corresponds to the particular case $z=1/2$ of the next more general result.

Lemma 1. *For every nonnegative integer k, and every complex number z, with $\Re z\leq 1/2$, we have*

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n(n+1)\cdots(n+k)}\left(\frac{z}{1-z}\right)^n=\frac{z^{-k}}{k!}\left(-\operatorname{Log}(1-z)-\sum_{j=1}^{k}\frac{z^j}{j}\right).$$

Proof. The case $k=0$ follows from the well-known power series expansion of $w\mapsto \operatorname{Log}(1+w)$. So, we will assume that $k>0$. Noting that

$$\frac{k!}{n(n+1)\cdots(n+k)}=\frac{\Gamma(k+1)\Gamma(n)}{\Gamma(n+k)}=\beta(k+1,n)=\int_0^1 t^k(1-t)^{n-1}dt$$

we see that for $|u|\leq 1$ we have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{k!}{n(n+1)\cdots(n+k)}u^n&=\int_0^1 t^k\left(\sum_{n=1}^{\infty}((1-t)u)^{n-1}\right)u\,dt\\&=\int_0^1\frac{t^k u}{1-u(1-t)}dt\end{aligned}$$

In what follows, Log is the principal determination of the logarithm defined in the domain $\mathbb{C}\setminus(-\infty,0]$, and we will write $\int_{[0,z]}f(\xi)d\xi$ to denote the path integration of f on the line segment $[0,z]$ in the complex plane $\mathbb{C}$.

Now, note that for $\Re z\leq\frac{1}{2}$ we have $\left|\frac{z}{1-z}\right|\leq 1$ and consequently, taking $u=-z/(1-z)$ in the previous formula we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{k!(-1)^{n-1}}{n(n+1)\cdots(n+k)}\left(\frac{z}{1-z}\right)^n&=\int_0^1\frac{t^k}{1-zt}z\,dt\\&=\frac{1}{z^k}\int_0^1\frac{(zt)^k}{1-zt}zdt\\&=\frac{1}{z^k}\int_{[0,z]}\frac{\xi^k}{1-\xi}d\xi\end{aligned}$$

But $\frac{\xi^k}{1-\xi} = \frac{1}{1-\xi} - (1 + \xi + \xi^2 + \cdots + \xi^{k-1})$, hence

$$\sum_{n=1}^{\infty} \frac{k!(-1)^{n-1}}{n(n+1)\cdots(n+k)} \left(\frac{z}{1-z}\right)^n = \frac{1}{z^k} \left(-\operatorname{Log}(1-z) - \sum_{j=1}^{k} \frac{z^j}{j} \right)$$

and the lemma follows. $\square$

Remark. Choosing $z = 1/(k+1)$, when $k \geq 1$ we obtain

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(n+1)\cdots(n+k)k^n} = \frac{(k+1)^k}{k!} \left(\ln\left(1 + \frac{1}{k}\right) - \sum_{j=1}^{k} \frac{1}{j(k+1)^j} \right).$$

V4-**2.** *Proposed by the editor.* Show that

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(n-2k)^{n+2}=\frac{2^n n(n+2)!}{6}.$$

Solution V4-2, by Omran Kouba, *Higher Institute for Applied Sciences and Technology, Damascus, Syria.*

We have the following result.

Lemma 2. *For every polynomial $P\in\mathbb{C}[X]$ with $\deg P\leq n+2$ and every $\tau\in\mathbb{C}$ we have*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}P(X-\tau k)=\tau^nP^{(n)}(X)-\frac{n}{2}\tau^{n+1}P^{(n+1)}(X)+\frac{n+3n^2}{24}\tau^{n+2}P^{(n+2)}(X)$$

Proof. The function $z\mapsto\frac{e^z-1}{z}$ is entire, and in the neighborhood of 0 we have

$$\frac{e^z-1}{z}=1+\frac{z}{2}+\frac{z^2}{6}+\mathcal{O}(z^3)$$

Hence

$$\begin{aligned}\left(\frac{e^z-1}{z}\right)^n&=\left(1+\frac{z}{2}+\frac{z^2}{6}+\mathcal{O}(z^3)\right)^n.\\&=1+n\left(\frac{z}{2}+\frac{z^2}{6}\right)+\frac{n(n-1)}{2}\left(\frac{z}{2}+\frac{z^2}{6}\right)^2+\mathcal{O}(z^3)\\&=1+\frac{n}{2}z+\frac{n+3n^2}{24}z^2+\mathcal{O}(z^3)\end{aligned}$$

This implies that

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}e^{kz}=(e^z-1)^n=z^n+\frac{n}{2}z^{n+1}+\frac{n+3n^2}{24}z^{n+2}+\mathcal{O}(z^{n+3})$$

Comparing the coefficients of z^j for $0\leq j\leq n+2$ on both sides we obtain

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{k^j}{j!}=\begin{cases}0 & \text{if } 0\leq j<n,\\ 1 & \text{if } j=n,\\ \frac{n}{2} & \text{if } j=n+1,\\ \frac{n+3n^2}{24} & \text{if } j=n+2,\end{cases}$$

Now, given a polynomial $Q\in\mathbb{C}[X]$ with $\deg Q\leq n+2$, we have $Q(X)=\sum_{j=0}^{n+2}\frac{Q^{(j)}(0)}{j!}X^j$. Thus

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}Q(k)=Q^{(n)}(0)+\frac{n}{2}Q^{(n+1)}(0)+\frac{n+3n^2}{24}Q^{(n+2)}(0).$$

And the lemma follows by applying this to $Q(X)=P(z-\tau X)$. □

Now, using the lemma with $P(X)=X^{n+2}$ and $\tau=2$ we get

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(X-2k)^{n+2}=(n+2)!2^{n-1}\left((X-n)^2+\frac{n}{3}\right),$$

and the desired formula follows by substituting $X=n$. □

V4-**3.** *Proposed by the editor.* Let n be a nonnegative integer, m a positive integer and $x \in \mathbb{C}$. Show that for the values of n, m, x for which the denominators do not vanish, the following identity holds:

$$\sum_{k=0}^{n} (-1)^k \frac{\binom{n}{k}\binom{x}{m-k}}{(m+n-k)\binom{x+n}{m+n-k}} = \frac{1}{m}\delta_{n0},$$

where $\delta_{n0} = \begin{cases} 1, & n=0 \\ 0, & n \neq 0 \end{cases}$ is Kronecker's delta.

Solution V4-3, by Omran Kouba, *Higher Institute for Applied Sciences and Technology, Damascus, Syria.*

Note that for $n = 0$ the desired identity is trivially true. So, let us suppose that $n > 0$. Clearly we have

$$\begin{aligned}\frac{\binom{x}{m-k}}{(m+n-k)\binom{x+n}{m+n-k}} &= \frac{x(x-1)\cdots(x-m+k+1)}{(m-k)!} \frac{(n+m-k-1)!}{(x+n)(x+n-1)\cdots(x-m+k+1)} \\ &= \frac{(n-1)!}{(x+n)\cdots(x+1)} \binom{n+m-k-1}{n-1}\end{aligned}$$

Thus, if we define the polynomial $Q(X)$ by the formula $Q(X) = \binom{n+m-X-1}{n-1}$, then clearly we have $\deg Q = n-1$ and

$$\sum_{k=0}^{n} (-1)^k \frac{\binom{n}{k}\binom{x}{m-k}}{(m+n-k)\binom{x+n}{m+n-k}} = \frac{(n-1)!}{(x+n)\cdots(x+1)} \sum_{k=0}^{n} (-1)^k \binom{n}{k} Q(k) = 0,$$

where, for the last equality, we used the Lemma 2 of the solution to Problem V4-2. The desired conclusion follows. $\square$

V4-**4.** *Proposed by Moubinool Omarjee, Lycée Henri IV, Paris, France.*

For $n \in \mathbb{Z}$ and $N \in \mathbb{N}$, let

$$a_n = \int_0^1 \int_0^1 e^{-|x-y|+2in\pi(x-y)} dx\, dy \quad \text{and} \quad S_N = \sum_{(m,n)\in I_N} a_m a_n$$

where $I_N = \{(m,n) \in \mathbb{Z}^2 : |m| \geq N \text{ or } |n| \geq N\}$. Evaluate $\lim_{N\to\infty}(NS_N)$, if it exists.

Solution V4-4, by Omran Kouba, *Higher Institute for Applied Sciences and Technology, Damascus, Syria.*

Note that

$$\begin{aligned}
a_n &= \int_0^1 \int_0^y e^{-|x-y|+2i\pi n(x-y)} dx\, dy + \int_0^1 \int_y^1 e^{-|x-y|+2i\pi n(x-y)} dx\, dy \\
&= \int_0^1 \int_0^y e^{-|x-y|+2i\pi n(x-y)} dx\, dy + \int_0^1 \int_0^x e^{-|x-y|+2i\pi n(x-y)} dy\, dx \\
&= \int_0^1 \int_0^y e^{-|x-y|+2i\pi n(x-y)} dx\, dy + \int_0^1 \int_0^y e^{-|x-y|+2i\pi n(y-x)} dx\, dy \\
&= 2\int_0^1 \int_0^y e^{x-y} \cos(2\pi n(x-y)) dx\, dy = 2\int_0^1 \left(\int_0^y e^{-t}\cos(2\pi nt)dt\right) dy
\end{aligned}$$

where we used the change of variables $t = y - x$ in the last inner integral. Thus,

$$\begin{aligned}
a_n &= \left[2(y-1)\left(\int_0^y e^{-t}\cos(2\pi nt)dt\right)\right]_{y=0}^{y=1} - 2\int_0^1 (y-1)e^{-y}\cos(2\pi ny)dy \\
&= 2\int_0^1 (1-y)e^{-y}\cos(2\pi ny)dy = 2\Re(c_n), \qquad (1)
\end{aligned}$$

here c_n is defined as follows:

$$\begin{aligned}
c_n &= \int_0^1 (1-y)e^{-y}\, e^{-2in\pi y} dy = \int_0^1 (1-y)e^{(-1-2i\pi n)y} dy \\
&= \left[\frac{(1-y)e^{(-1-2i\pi n)y}}{-1-2i\pi n}\right]_{y=0}^{y=1} + \frac{1}{-1-2i\pi n}\int_0^1 e^{(-1-2i\pi n)y} dy \\
&= \frac{1}{1+2i\pi n} + \frac{e^{-1}-1}{(1+2i\pi n)^2}. \qquad (2)
\end{aligned}$$

Thus

$$\begin{aligned}
a_n &= \frac{2}{1+4\pi^2 n^2} + 2(e^{-1}-1)\frac{1-4\pi^2 n^2}{(1+4\pi^2 n^2)^2} \\
&= \frac{4-2e^{-1}}{1+4\pi^2 n^2} - \frac{4(1-e^{-1})}{(1+4\pi^2 n^2)^2}. \qquad (3)
\end{aligned}$$

This shows that $a_n = a_{-n} > 0$ for every integer n, and $a_n = \mathcal{O}(1/n^2)$. So, we can define

$$A = \sum_{n\in\mathbb{Z}} a_n, \quad \text{and} \quad R_N = \sum_{|n|\geq N} a_n = 2\sum_{n=N}^{\infty} a_n.$$

Further, the double series $\sum_{m,n} a_m a_n$ is convergent and

$$S_N = \sum_{(m,n)\in\mathbb{Z}^2} a_m a_n - \sum_{|m|<N,|n|<N} a_m a_n = A^2 - (A - R_N)^2 = (2A - R_N)R_N. \tag{4}$$

Now note that, according to (3), we have

$$a_n - \frac{2 - e^{-1}}{2\pi^2 n(n+1)} = \mathcal{O}\left(\frac{1}{n^3}\right),$$

so,

$$\sum_{n=N}^{\infty} \left(a_n - \frac{2 - e^{-1}}{2\pi^2 n(n+1)}\right) = \mathcal{O}\left(\frac{1}{N^2}\right),$$

or equivalently

$$R_N = 2\sum_{n=N}^{\infty} a_n = \frac{2 - e^{-1}}{\pi^2 N} + \mathcal{O}\left(\frac{1}{N^2}\right) \tag{5}$$

On the other hand, the definition of c_n in formula (2) shows that c_n is the exponential Fourier coefficient $C_n(f)$ of the 1-periodic function f defined on $(0,1)$ by $f(t) = (1-t)e^{-t}$ with $f(0) = 1/2$. Using, Dirichlet's test, we know that, for every $t \in [0,1)$. we have

$$f(t) = \sum_{n\in\mathbb{Z}} c_n e^{2i\pi nt}$$

In particular, $\frac{1}{2} = f(0) = \lim_{N\to\infty} \sum_{-N}^{N} c_n$. Taking, real parts and recalling that $a_n = 2\Re c_n$, we obtain $A = 1$. Combining $A = 1$ and (5) with (4) we obtain

$$S_N = \frac{2(2 - e^{-1})}{\pi^2 N} + \mathcal{O}\left(\frac{1}{N^2}\right)$$

and consequently $\lim\limits_{N\to\infty} (NS_N) = \dfrac{2(2 - e^{-1})}{\pi^2}$. $\square$

V4-**5.** *Proposed by Serafeim Tsipelis, Ioannina, Greece.* Show that

$$\int_0^{\pi/2} x\log(1-\cos x)\,dx = \frac{35}{16}\zeta(3) - \frac{\pi^2\log 2}{8} - \pi G,$$

where G is the Catalan constant and ζ is the Riemann zeta function.

Solution V4-5, by Omran Kouba, *Higher Institute for Applied Sciences and Technology, Damascus, Syria.*

Our starting point is the well-known power series expansion

$$-\operatorname{Log}(1-z) = \sum_{n=1}^{\infty}\frac{z^n}{n}$$

where Log is the principal determination of the logarithm defind in the domain $\mathbb{C}\setminus(-\infty,0]$. Using Abel's transformation, it is easy to see that $\sum_{n=1}^{\infty}\frac{e^{inx}}{n}$ is convergent for every $x\in(0,2\pi)$, and using Abel's theorem we conclude that, for every $x\in(0,2\pi)$,

$$-\operatorname{Log}(1-e^{ix}) = \sum_{n=1}^{\infty}\frac{e^{inx}}{n}$$

Taking real parts we find that

$$-\log\left|1-e^{ix}\right|^2 = \sum_{n\in\mathbb{Z}^*}^{\infty}\frac{e^{inx}}{|n|}$$

Finally, for every $x\in(0,2\pi)$,

$$-\log(2(1-\cos x)) = \sum_{n\in\mathbb{Z}^*}^{\infty}\frac{e^{inx}}{|n|}$$

So, if f is the 2π-periodic function defined by $f(x)=\log(2(1-\cos x))$ on $(0,2\pi)$, then clearly f belongs to $L^2(T)$ and its exponential Fourier coefficients are given by $C_n(f)=-1/|n|$ for $n\neq 0$ and $C_0(f)=0$.

On the other hand, let g be the 2π-periodic function defined on $(0,2\pi)$ by $g(x)=x$ if $0\le x\le\pi/2$ and $g(x)=0$ otherwise. Clearly $g\in L^2(T)$, and if $(C_n(g))_{n\in\mathbb{Z}}$ are its exponential Fourier coefficients then $C_0(g)=\frac{\pi}{16}$ and

$$\begin{aligned}
C_n(g) &= \frac{1}{2\pi}\int_0^{\pi/2} xe^{-inx}dx = \frac{1}{2\pi}\left(\frac{\pi(-i)^n}{-2in} + \frac{1}{in}\int_0^{\pi/2} e^{-inx}dx\right)\\
&= \frac{(-i)^{n-1}}{4n} + \frac{(-i)^n-1}{2\pi n^2}
\end{aligned}$$

Using Parseval's formula:

$$\frac{1}{2\pi}\int_0^{2\pi} f(x)\overline{g(x)}\,dx = \sum_{n\in\mathbb{Z}} C_n(f)\overline{C_n(g)}$$

we obtain

$$\begin{aligned}
\int_0^{\pi/2} x\log(2(1-\cos x))dx &= -\sum_{n\in\mathbb{Z}^*}\left(\frac{\pi i^{n-1}}{2n\,|n|}+\frac{i^n-1}{n^2\,|n|}\right)\\
&= -\sum_{n=1}^{\infty}\left(\frac{\pi(1-(-1)^n)i^{n-1}}{4n^2}+\frac{(1+(-1)^n)i^n-2}{n^3}\right)\\
&= 2\zeta(3)-\pi\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}+\frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3}\\
&= 2\zeta(3)-\pi G+\frac{1}{4}\cdot\frac{3}{4}\zeta(3)\\
&= \frac{35}{16}\zeta(3)-\pi G
\end{aligned}$$

or

$$\int_0^{\pi/2}(x\log 2+x\log(1-\cos x))dx=\frac{35}{16}\zeta(3)-\pi G$$

and finally,

$$\int_0^{\pi/2} x\log(1-\cos x)\,dx=\frac{35}{16}\zeta(3)-\frac{\pi^2\log 2}{8}-\pi G$$

which is the desired result. $\square$

V4-**6.** *Proposed by Serafeim Tsipelis, Ioannina, Greece.* Evaluate $\displaystyle\int_0^{+\infty} \frac{\log(\cos^2 x)}{1+e^{2x}}\,dx.$

Solution V4-6, by Omran Kouba, *Higher Institute for Applied Sciences and Technology, Damascus, Syria.*

Let us denote the considered integral by I. Clearly we have

$$I = \frac{1}{2}\int_0^{+\infty} \frac{\log(\cos^2(x/2))}{1+e^{x}}\,dx = \frac{1}{4}\int_{-\infty}^{+\infty} \frac{\log(\cos^2(x/2))}{1+e^{|x|}}\,dx.$$

Now, consider the functions f and g defined by

$$f(x) = \frac{1}{1+e^{|x|}} \quad \text{and} \quad g(x) = \log(\cos^2(x/2))$$

Clearly g is a square-integrable 2π-periodic function. Moreover, for every $x \in (-\pi, \pi)$ we have

$$\begin{aligned} g(x) &= -2\log 2 + 2\log\left|1+e^{ix}\right| = -2\log 2 + 2\Re(\mathrm{Log}(1+e^{ix})) \\ &= -2\log 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\left(e^{inx} + e^{-inx}\right) \end{aligned}$$

This proves that the exponential Fourier coefficients $(C_n(g))_{n\in\mathbb{Z}}$ of g are given by

$$C_0(g) = -2\log 2, \quad \text{and } C_n(g) = \frac{(-1)^{n-1}}{|n|} \text{ for } n \neq 0.$$

On the other hand, it is clear that we define a 2π-periodic function F by setting

$$F(x) = \sum_{k\in\mathbb{Z}} f(x-2\pi k) = \sum_{k\in\mathbb{Z}} \frac{1}{1+e^{|x-2\pi k|}},$$

and it is easy to see that F is a continuous function on $[0, 2\pi]$, because the series defining F is uniformly convergent on this interval. Moreover, if $f_k(x) = f(x-2\pi k)$ then clearly

$$\int_0^{2\pi} (f_k(x))^2\,dx \le \int_{2\pi k}^{2\pi(k+1)} e^{-2|t|}\,dt = \sinh(2\pi)e^{-2\pi|2k+1|}$$

Thus, the series $\sum_{k\in\mathbb{Z}} \|f_k\|_{L^2(T)}$ is convergent, and the series defining F converges normally in $L^2(T)$; the space of square integrable 2π periodic functions. Thus $F \in L^2(T)$, using Parseval's identity, we conclude that

$$\frac{1}{2\pi}\int_0^{2\pi} F(x)\overline{g(x)}\,dx = \sum_{n\in\mathbb{Z}} C_n(F)\overline{C_n(g)}$$

But,

$$\begin{aligned} \int_0^{2\pi} F(x)\overline{g(x)}\,dx &= \lim_{n\to\infty}\sum_{k=-n}^{n-1}\int_0^{2\pi} f(x-2\pi k)g(x-2\pi k)\,dx \\ &= \lim_{n\to\infty}\int_{-2\pi n}^{2\pi n} f(x)g(x)\,dx = \int_{\mathbb{R}} f(x)g(x) = 4I \end{aligned}$$

Also, in a similar way, we have

$$2\pi C_n(F) = \int_0^{2\pi} F(x)e^{-inx}\,dx = \int_{\mathbb{R}} f(x)e^{-inx}\,dx = \widehat{f}(n)$$

where $\widehat{f}$ is the Fourier transform of f.

For $w \in \mathbb{R}$ we have

$$\widehat{f}(w) = \int_{\mathbb{R}} \frac{e^{-iwx}}{1+e^{|x|}}\,dx = 2\int_0^{\infty} \frac{\cos(wx)}{1+e^x}\,dx$$

So

$$\left|\widehat{f}(w) - 2\sum_{p=1}^{q}(-1)^{p-1}\int_0^{\infty} e^{-px}\cos(wx)\,dx\right| \le 2\int_0^{\infty} e^{-(q+1)x}\,dx = \frac{2}{q+1}$$

consequently

$$\widehat{f}(w) = \sum_{p=1}^{\infty}(-1)^{p-1}\frac{2p}{p^2+w^2}$$

because $\int_0^{\infty} e^{-px}\cos(wx)\,dx = \frac{p}{p^2+w^2}$. In particular, $\widehat{f}(0) = 2\log 2$. Combining the above results we obtain

$$4I = -4\log^2 2 + 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\sum_{p=1}^{\infty}(-1)^{p-1}\frac{2p}{p^2+n^2}\right)$$

Or, $I = -\log^2 2 + J$ with

$$J = \sum_{n=1}^{\infty}\left(\sum_{p=1}^{\infty}\frac{(-1)^{p+n}p}{n(n^2+p^2)}\right)$$

Now, this double series is not absolutely convergent, so we must be careful. First, exchanging the roles of p and n we have

$$J = \sum_{p=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{(-1)^{p+n}n}{p(n^2+p^2)}\right)$$

Now, using the properties of convergent alternating we have

$$\sum_{n=1}^{\infty}\frac{(-1)^{p+n}n}{p(n^2+p^2)} = \sum_{n=1}^{q-1}\frac{(-1)^{p+n}n}{p(n^2+p^2)} + R_q(p),$$

with

$$R_q(p) = \frac{(-1)^p}{p}\sum_{n=q}^{\infty}\frac{(-1)^n n}{n^2+p^2} \quad \text{and} \quad |R_q(p)| \le \frac{1}{p}\cdot\frac{q}{p^2+q^2}$$

Thus

$$J = sum_{n=1}^{q-1}\left(\sum_{p=1}^{\infty}\frac{(-1)^{p+n}n}{p(n^2+p^2)}\right) + \epsilon_q$$

with $\epsilon_q = \sum_{p=1}^{\infty} R_q(p)$. But

$$\epsilon_q \le \sum_{p=1}^{\infty}\frac{q}{p(p^2+q^2)}$$

Now, since $\frac{q}{p(p^2+q^2)} \le \frac{1}{2p^2}$ for every q, the series $\sum 1/(2p^2) < +\infty$ and $\lim_{q\to\infty}\frac{q}{p(p^2+q^2)} = 0$ for every p, we conclude that $\lim_{q\to\infty}\epsilon_q = 0$, So, letting p tend to $+\infty$ we conclude that

$$J = \sum_{n=1}^{\infty}\left(\sum_{p=1}^{\infty}\frac{(-1)^{p+n}n}{p(n^2+p^2)}\right)$$

Taking the sum of the two expressions of J we obtain

$$2J = \sum_{n=1}^{\infty}\left(\sum_{p=1}^{\infty}\frac{(-1)^{p+n}}{n^2+p^2}\left(\frac{n}{p}+\frac{p}{n}\right)\right) = \sum_{n=1}^{\infty}\left(\sum_{p=1}^{\infty}\frac{(-1)^{p+n}}{np}\right) = (-\log 2)^2 = \log^2 2.$$

We conclude that $I = -\log^2 2 + J = -\frac{\log^2 2}{2}$. which is the desired conclusion. $\square$

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V4-**8.** (∗) *Proposed by Konstantinos Tsouvalas,* University of Athens, Athens, Greece.

1. Show that $\left(\frac{2}{3}\right)^n \sum_{k=0}^{\lfloor n/3 \rfloor} \binom{n}{k} 2^{-k} = \frac{1}{2}$ without using probabilistic methods.

2. Can we find a better approximation of the quantity $\left(\frac{2}{3}\right)^n \sum_{k=0}^{\lfloor n/3 \rfloor} \binom{n}{k} 2^{-k}$ than

$$\left(\frac{2}{3}\right)^n \sum_{k=0}^{\lfloor n/3 \rfloor} \binom{n}{k} 2^{-k} = \frac{1}{2} + o(1)\ ?$$

$\lfloor \cdot \rfloor$ denotes the integer part.

Solution V4-8, by Omran Kouba, *Higher Institute for Applied Sciences and Technology, Damascus, Syria.*

For a positive integer n, an integer k with $0 \leq k \leq n$, and real $p \in (0,1)$ we consider

$$U(k,p,n) = (n-k)\binom{n}{k} \int_0^q t^{n-k-1}(1-t)^k dt$$

where $q = 1-p$. Now, an integration by parts shows that, for $0 < k < n$ we have

$$\begin{aligned} U(k,p,n) &= \left[\binom{n}{k} t^{n-k}(1-t)^k\right]_{t=0}^q + k\binom{n}{k} \int_0^q t^{n-k}(1-t)^{k-1} d \\ &= \binom{n}{k} p^k q^{n-k} + U(k-1,p,n), \end{aligned}$$

and clearly $U(0,p,n) = q^n$, thus

$$U(m,p,n) = \sum_{k=0}^{m} \binom{n}{k} p^k q^{n-k}$$

We are interested in $S_n = U(\lfloor n/3 \rfloor, 1/3, n)$, and according to what we have proved we have

$$\begin{aligned} S_{3n+1} &= (3n+1)\binom{3n}{n} \int_0^{2/3} (t^2(1-t))^n dt \\ S_{3n+2} &= \frac{(3n+2)(3n+1)}{2n+1}\binom{3n}{n} \int_0^{2/3} (t^2(1-t))^n t dt \\ S_{3n+3} &= \frac{3(3n+2)(3n+1)}{2n+1}\binom{3n}{n} \int_0^{2/3} (t^2(1-t))^n t(1-t) dt \end{aligned}$$

Now, the treatment of the three integrals is similar and standard. Let $I_n = \int_0^{2/3} (t^2(1-t))^n g(t) dt$, with $g(t) = 1$, $g(t) = t$ or $g(t) = t(1-t)$ according to the considered case. since $t \mapsto t^2 - t^3$ attains its maximum on $[0, 2/3]$ at $t = 2/3$, the change of variables $t = 2(1-u)/3$ shows that

$$I_n = \frac{2}{3}\left(\frac{4}{27}\right)^n \int_0^1 (1 - 3u^2 + 2u^3)^n g\left(\frac{2(1-u)}{3}\right) du$$

Thus

$$S_{3n+1} = \frac{2(3n+1)}{3}\left(\frac{4}{27}\right)^n \binom{3n}{n}\int_0^1 (1-3u^2+2u^3)^n du$$
$$S_{3n+2} = \frac{4(3n+2)(3n+1)}{9(2n+1)}\left(\frac{4}{27}\right)^n \binom{3n}{n}\int_0^1 (1-3u^2+2u^3)^n(1-u) du$$
$$S_{3n+3} = \frac{4(3n+2)(3n+1)}{9(2n+1)}\left(\frac{4}{27}\right)^n \binom{3n}{n}\int_0^1 (1-3u^2+2u^3)^n(1-u)(1+2u) du$$

Now, using the well-known expansion

$$n! = \sqrt{2\pi n}\, n^n e^{-n}\left(1+\frac{1}{12n}+O\left(\frac{1}{n^2}\right)\right)$$

we conclude that
$$\left(\frac{4}{27}\right)^n \binom{3n}{n} = \frac{3}{2\sqrt{3\pi n}} - \frac{7}{48n\sqrt{3\pi n}} + +O\left(\frac{1}{n^{5/2}}\right)$$

and consequently

$$S_{3n+1} = \sqrt{\frac{3n}{\pi}}\left(1+\frac{17}{72n}+O\left(\frac{1}{n^2}\right)\right)J_n^{(1)}$$
$$S_{3n+2} = \sqrt{\frac{3n}{\pi}}\left(1+\frac{29}{72n}+O\left(\frac{1}{n^2}\right)\right)J_n^{(2)}$$
$$S_{3n+3} = \sqrt{\frac{3n}{\pi}}\left(1+\frac{29}{72n}+O\left(\frac{1}{n^2}\right)\right)J_n^{(3)}$$

Where
$$J_n^{(k)} = \int_0^1 (1-3u^2+2u^3)^n q_k(u) du$$
with $q_1(u) = 1$, $q_2(u) = 1-u$ and $q_3(u) = (1-u)(1+2u)$.

Now, the treatment of the integrals $J_n^{(k)}, k = 1, 2, 3$ is standard. we will follow F.W.J. Olver [1][Chapter 3, §8].

The function $u \mapsto p(u) = -\ln(1-3u^2+2u^3)$ defines a strictly increasing bijection from $[0, 1)$ onto $[0, +\infty)$, so we may consider its inverse function $\varphi = p^{-1}$, which is analytic on the interval $(0, 1)$, and the change of variables $u = \varphi(v)$ shows that

$$J_n^{(k)} = \int_0^\infty e^{-nv} q_k(\varphi(v))\varphi'(v) dv$$

Now, starting from the series expansion

$$p(u) = 3u^2 - 2u^3 + \frac{9}{2}u^4 - 6u^5 + O(u^6)$$

we can find the following asymptotic expansions of φ

$$\varphi(v) = \frac{1}{\sqrt{3}}\sqrt{v} + \frac{1}{9}v - \frac{17}{108\sqrt{3}}v\sqrt{v} - \frac{11}{486}v^2 + O(v^{5/2})$$
$$\varphi'(v) = \frac{1}{2\sqrt{3v}} + \frac{1}{9} - \frac{17}{72\sqrt{3}}\sqrt{v} - \frac{11}{243}v + O(v^{3/2})$$

$$\begin{aligned}
J_n^{(1)} &= \int_0^\infty e^{-nv}\left(\frac{1}{2\sqrt{3v}}+\frac{1}{9}-\frac{17}{72\sqrt{3}}\sqrt{v}-\frac{11}{243}v\right)dv+O\left(\frac{1}{n^{5/2}}\right)\\
&=\frac{1}{2}\sqrt{\frac{\pi}{3n}}+\frac{1}{9n}-\frac{17}{144n}\sqrt{\frac{\pi}{3n}}+O\left(\frac{1}{n^2}\right)
\end{aligned}$$

Similarly,

$$q_1(\varphi(v))\varphi'(v)=\frac{1}{2\sqrt{3v}}-\frac{1}{18}-\frac{29}{72\sqrt{3}}\sqrt{v}+\frac{23}{486}v+O(v^{3/2})$$

Thus,

$$\begin{aligned}
J_n^{(2)} &= \int_0^\infty e^{-nv}\left(\frac{1}{2\sqrt{3v}}-\frac{1}{18}-\frac{29}{72\sqrt{3}}\sqrt{v}+\frac{23}{486}v\right)dv+O\left(\frac{1}{n^{5/2}}\right)\\
&=\frac{1}{2}\sqrt{\frac{\pi}{3n}}-\frac{1}{18n}-\frac{29}{144n}\sqrt{\frac{\pi}{3n}}+O\left(\frac{1}{n^2}\right).
\end{aligned}$$

And

$$q_2(\varphi(v))\varphi'(v)=\frac{1}{2\sqrt{3v}}+\frac{5}{18}-\frac{29}{72\sqrt{3}}\sqrt{v}-\frac{139}{486}v+O(v^{3/2})$$

Thus,

$$\begin{aligned}
J_n^{(3)} &= \int_0^\infty e^{-nv}\left(\frac{1}{2\sqrt{3v}}+\frac{5}{18}-\frac{29}{72\sqrt{3}}\sqrt{v}-\frac{139}{486}v\right)dv+O\left(\frac{1}{n^{5/2}}\right)\\
&=\frac{1}{2}\sqrt{\frac{\pi}{3n}}+\frac{5}{18n}-\frac{29}{144n}\sqrt{\frac{\pi}{3n}}+O\left(\frac{1}{n^2}\right).
\end{aligned}$$

We conclude that

$$\begin{aligned}
S_{3n+1}&=\frac{1}{2}+\frac{1}{3\sqrt{3\pi n}}+\frac{17}{216n\sqrt{3\pi n}}+O\left(\frac{1}{n^2}\right)\\
S_{3n+2}&=\frac{1}{2}-\frac{1}{6\sqrt{3\pi n}}-\frac{29}{432n\sqrt{3\pi n}}+O\left(\frac{1}{n^2}\right)\\
S_{3n+3}&=\frac{1}{2}+\frac{5}{6\sqrt{3\pi n}}+\frac{145}{432n\sqrt{3\pi n}}+O\left(\frac{1}{n^2}\right)
\end{aligned}$$

and we are done. □

References

[1] F.W.J. Olver, *Asymptotics and Special Functions*, Academic Press, New York, London, 1974.

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