The Inequality Forum

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Happy New Year 2017

Ebook Written by:

TIF Community

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Problems

PROBLEM 1 (All russian olympiad 2016, Day 2, grade 9, P8). Let a, b, c, d be positive $a+b+c+d=3$. Prove that

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2} \le \frac{1}{a^2b^2c^2d^2}$$

PROBLEM 2 (All russian olympiad 2016, Day 2, grade 11, P7). Let a, b, c, d be positive real numbers such that $a+b+c+d=3$. Prove that

$$\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}+\frac{1}{d^3} \le \frac{1}{a^3b^3c^3d^3}$$

PROBLEM 3 (Azerbaijan BMO TST 2016). Let a, b, c be non-negative real numbers. Prove that

$$\begin{aligned} 3\left(a^2+b^2+c^2\right) &\ge (a+b+c)\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)+(a-b)^2+(b-c)^2+(c-a)^2 \\ &\ge (a+b+c)^2 \end{aligned}$$

PROBLEM 4 (Azerbaijan Junior Mathematical Olympiad 2016). Let x, y, z be real numbers. Prove that

$$\sqrt{x^2+\frac{1}{y^2}}+\sqrt{y^2+\frac{1}{z^2}}+\sqrt{z^2+\frac{1}{x^2}} \ge 3\sqrt{2}$$

PROBLEM 5 (2016 China Western Mathematical Olympiad). Let $a_1, a_2, \ldots, a_n$ be non-negative real numbers, $S_k = \sum\limits_{i=1}^{k} a_i$ with $1 \le k \le n$. Prove that

$$\sum_{i=1}^{n}\left(a_iS_i\sum_{j=i}^{n}a_j^2\right) \le \sum_{i=1}^{n}(a_iS_i)^2$$

PROBLEM 6 (Croatia Team Selection Test 2016). Let $n \ge 1$ and $x_1, \ldots, x_n \ge 0$. Prove that

$$\left(x_1+\frac{x_2}{2}+\ldots+\frac{x_n}{n}\right)(x_1+2x_2+\ldots+nx_n) \le \frac{(n+1)^2}{4n}(x_1+x_2+\ldots+x_n)^2$$

PROBLEM 7 (EGMO 2016). Let n be an odd positive integer, and let $x_1, x_2, \cdots, x_n$ be non-negative real numbers. Show that

$$\min_{i=1,\ldots,n}\left(x_i^2+x_{i+1}^2\right) \leq \max_{j=1,\ldots,n}\left(2x_jx_{j+1}\right)$$

where $x_{n+1} = x_1$.

PROBLEM 8 (Team Selection Test for EGMO). Prove that

$$x^4y+y^4z+z^4x+xyz\left(x^3+y^3+z^3\right) \geq (x+y+z)(3xyz-1)$$

for all positive real numbers x, y, z.

PROBLEM 9 (Korea National Olympiad Final Round 2016). If x, y, z satisfies $x^2+y^2+z^2=1$, find the maximum possible value of

$$\left(x^2-yz\right)\left(y^2-zx\right)\left(z^2-xy\right)$$

PROBLEM 10 (Hong Kong Team Selection Test 2016). Let a, b, c be positive real numbers satisfying $abc = 1$. Determine the smallest possible value of

$$\frac{a^3+8}{a^3(b+c)}+\frac{b^3+8}{b^3(a+c)}+\frac{c^3+8}{c^3(b+a)}$$

PROBLEM 11 (IMC 2016). Let n be a positive integer. Also let $a_1, a_2, \ldots, a_n$, $b_1, b_2, \ldots, b_n$ be real numbers such that $a_i + b_i > 0$ for $i = 1, 2, \ldots, n$. Prove that

$$\sum_{i=1}^{n}\frac{a_ib_i-b_i^2}{a_i+b_i} \leq \frac{\sum\limits_{i=1}^{n}a_i\cdot\sum\limits_{i=1}^{n}b_i-\left(\sum\limits_{i=1}^{n}b_i\right)^2}{\sum\limits_{i=1}^{n}(a_i+b_i)}$$

PROBLEM 12 (India International Mathematical Olympiad Training Camp 2016). Let a, b, c, d be real numbers satisfying $|a|, |b|, |c|, |d| > 1$ and $abc + abd + acd + bcd + a + b + c + d = 0$. Prove that

$$\frac{1}{a-1}+\frac{1}{b-1}+\frac{1}{c-1}+\frac{1}{d-1} > 0$$

PROBLEM 13 (India Regional Mathematical Olympiad 2016). Let a, b, c be positive real numbers such that $a+b+c = 3$. Determine, with certainty, the largest possible value of the expression

$$\frac{a}{a^3+b^2+c}+\frac{b}{b^3+c^2+a}+\frac{c}{c^3+a^2+b}$$

Problem 14 (2016 India National Math Olympiad (3rd Round)). Let $a, b, c \in \mathbb{R}^+$ and $abc = 1$. Prove that

$$\frac{a+b}{(a+b+1)^2} + \frac{b+c}{(b+c+1)^2} + \frac{c+a}{(c+a+1)^2} \geq \frac{2}{a+b+c}$$

Problem 15 (Iran TST 2016). Let a, b, c, d be positive real numbers such that

$$\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} + \frac{1}{d+1} = 2.$$

Prove that

$$\sqrt{\frac{a^2+1}{2}} + \sqrt{\frac{b^2+1}{2}} + \sqrt{\frac{c^2+1}{2}} + \sqrt{\frac{d^2+1}{2}} \geq 3\left(\sqrt{a} + \sqrt{b} + \sqrt{c} + \sqrt{d}\right) - 8$$

Problem 16 (Team Selection Test for JBMO). Prove that

$$(x^4+y)(y^4+z)(z^4+x) \geq (x+y^2)(y+z^2)(z+x^2)$$

for all positive real numbers x, y, z satisfying $xyz \geq 1$.

Problem 17 (Junior Balkan MO 2016). Let a, b, c be positive real numbers. Prove that

$$\frac{8}{(a+b)^2+4abc} + \frac{8}{(b+c)^2+4abc} + \frac{8}{(a+c)^2+4abc} + a^2+b^2+c^2 \geq \frac{8}{a+3} + \frac{8}{b+3} + \frac{8}{c+3}$$

Problem 18 (Junior Balkan Team Selection Test 2016). Let $a, b, c \in \mathbb{R}^+$, prove that

$$\frac{2a}{\sqrt{3a+b}} + \frac{2b}{\sqrt{3b+c}} + \frac{2c}{\sqrt{3c+a}} \leq \sqrt{3(a+b+c)}$$

Problem 19 (Junior Balkan Team Selection Tests - Romania 2016). Let $a, b, c > 0$ and $abc \geq 1$. Prove that

$$\frac{1}{a^3+2b^3+6} + \frac{1}{b^3+2c^3+6} + \frac{1}{c^3+2a^3+6} \leq \frac{1}{3}$$

Problem 20 (Junior Balkan Team Selection Tests - Romania 2016). Let a, b, c be real numbers such that $a \geq b \geq 1 \geq c \geq 0$ and $a+b+c = 3$.

a) Prove that $2 \leq ab+bc+ca \leq 3$
b) Prove that

$$\frac{24}{a^3+b^3+c^3} + \frac{25}{ab+bc+ca} \geq 14$$

Problem 21 (Korea Winter Program Practice Test 2016). Let $x, y, z \geq 0$ be real numbers such that $(x+y-1)^2 + (y+z-1)^2 + (z+x-1)^2 = 27$.

Find the maximum and minimum of $x^4+y^4+z^4$

PROBLEM 22 (Macedonian National Olympiad 2016). Let $n \geq 3$ and $a_1, a_2, ..., a_n$ be real positive numbers such that

$$\frac{1}{1+a_1^4} + \frac{1}{1+a_2^4} + ... + \frac{1}{1+a_n^4} = 1$$

Prove that

$$a_1 a_2 ... a_n \geq (n-1)^{\frac{n}{4}}$$

PROBLEM 23 (Mediterranean Mathematics Olympiad 2016). Let a, b, c be positive real numbers with $a+b+c=3$. Prove that

$$\sqrt{\frac{b}{a^2+3}} + \sqrt{\frac{c}{b^2+3}} + \sqrt{\frac{a}{c^2+3}} \leq \frac{3}{2}\sqrt[4]{\frac{1}{abc}}$$

PROBLEM 24 (Middle European Mathematical Olympiad 2016). Let $n \geq 2$ be an integer, and let $x_1, x_2, \ldots, x_n$ be reals for which

(a) $x_j > -1$ for $j = 1, 2, \ldots, n$ and
(b) $x_1 + x_2 + \ldots + x_n = n$.
Prove that

$$\sum_{j=1}^{n} \frac{1}{1+x_j} \geq \sum_{j=1}^{n} \frac{x_j}{1+x_j^2}$$

PROBLEM 25 (Pan-African Mathematical Olympiad problems from 2016). Let x, y, z be positive real numbers such that $xyz = 1$. Prove that

$$\frac{1}{(x+1)^2+y^2+1} + \frac{1}{(y+1)^2+z^2+1} + \frac{1}{(z+1)^2+x^2+1} \leq \frac{1}{2}$$

PROBLEM 26 (2016 San Diego Math Olympiad). Let u, v, w be positive real numbers such that $u\sqrt{vw}+v\sqrt{wu}+w\sqrt{uv} \geq 1$. Find the smallest value of $u+v+w$.

PROBLEM 27 (2016 Selection round of Kiev team to UMO). Let $a, b, c > 0$ such that $a+b+c=3$, prove that

$$\frac{a^2}{a+b^2} + \frac{b^2}{b+c^2} + \frac{c^2}{c+a^2} \geq \frac{3}{2}$$

PROBLEM 28 (2016 Selection round of Kiev team to UMO). Let be positive real numbers x, y, z. Prove that

$$\frac{x^2}{xy+z} + \frac{y^2}{yz+x} + \frac{z^2}{xz+y} \geq \frac{(x+y+z)^3}{3(x^2(y+1)+y^2(z+1)+z^2(x+1)}$$

PROBLEM 29 (2016 Taiwan TST Round 2). Let $x, y > 0$ such that $x+y=1$. Prove that

$$\frac{x}{x^2+y^3} + \frac{y}{x^3+y^2} \leq 2\left(\frac{x}{x+y^2} + \frac{y}{x^2+y}\right)$$

PROBLEM 30 (Taiwan TST Round 3). Let $x, y, z > 0$ such that $x + y + z = 1$. Find the smallest k such that

$$\frac{x^2y^2}{1-z} + \frac{y^2z^2}{1-x} + \frac{z^2x^2}{1-y} \le k - 3xyz$$

PROBLEM 31 (Turkey EGMO TST 2016). For all $x, y, z > 0$. Prove that

$$x^4y + y^4z + z^4x + xyz\left(x^3 + y^3 + z^3\right) \ge (x + y + z)(3xyz - 1)$$

PROBLEM 32 (Turkey Team Selection Test 2016). Let $a, b, c \ge 0$ such that $a^2 + b^2 + c^2 \le 3$. Prove that

$$(a + b + c)(a + b + c - abc) \ge 2\left(a^2b + b^2c + c^2a\right)$$

PROBLEM 33 (2016 Turkmenistan Regional Math Olympia). If a, b, c are triangle sides. Prove that

$$\sqrt{\frac{a}{-a+b+c}} + \sqrt{\frac{b}{-b+c+a}} + \sqrt{\frac{c}{-c+a+b}} \ge 3$$

PROBLEM 34 (India Regional Mathematical Olympiad 2016). Let $a, b, c > 0$ such that

$$\frac{a}{1+a} + \frac{b}{1+b} + \frac{c}{1+c} = 1.$$

Prove that $abc \le \frac{1}{8}$.

PROBLEM 35 (Spain Mathematical Olympiad 2016). Let $n \ge 2$ an integer. Find the least value of γ such that for any positive real numbers $x_1, x_2, ..., x_n$ with $x_1 + x_2 + ... + x_n = 1$ and $y_1 + y_2 + ... + y_n = 1$ with $0 \le y_1, y_2, ..., y_n \le \frac{1}{2}$ the following inequality holds

$$x_1x_2...x_n \le \gamma\left(x_1y_1 + x_2y_2 + ... + x_ny_n\right)$$