Category theory via definitions and examples

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Abstract

This note is accessible to every two years undergraduated student. It is mostly based on my knowledge and from typical textbooks in the field like. To anyone who is able to understand every of example in this note then you have wasted a little time in your life. With those who do not understand the most then you should prepare more to comeback stronger. An with those who understand half of all examples but also get some perspectives on other examples then you are appropriate with this note. Thank you so much!

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Contents

1 Fundamental concepts

Definition 1.1. A **category** $\mathcal{C}$ contains a class of objects $\mathrm{Obj}(\mathcal{C})$ and set of morphisms between any ordered pair of objects $(A, B), \mathrm{Hom}(A, B)$, and the **composition** $\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$ denoted by:

$$(f, g) \mapsto gf \ \forall f \in \mathrm{Hom}(A, B), g \in \mathrm{Hom}(B, C)$$

These ingredients must satisfy the following axioms:

(1) Each morphism f has a unique **domain** and a unique **target**. Alternatively, Hom sets are pairwise disjoint.

(2) For each object A there is a special morphism, called **identity** of A, $1_A \in \text{Hom}(A, A)$ such that $f1_A = 1_A f = f$ whenever the composition is allowed.

(3) Composition is associative in the sense that $h(gf) = (hg)f$ whenever the composition of these morphism is possible.

If we keep all objects of $\mathcal{C}$ and reserves all directions of arrows then we have its opposite category $\mathcal{C}^{op}$. This construction has an advantage that it reduces half of our works in category theorem.
Note. There are examples of category that is not "***isomorphic***" to its opposite category.

Example 1.2 (**Sets**)**.** The category of all sets (not proper class), morphism are fucntions and compisition is taken in the usual sense.

Example 1.3 (**Grps**)**.** The category of all groups, morphisms are group homomorphisms and composition is the usual composition of functions.

Example 1.4 (**Rings**)**.** The category of all rings, morphisms are ring homomorphisms and composition is the usual composition of functions. If we require rings have unit element then we also restrict morphisms take unit to unit.

Example 1.5 (**Fields**)**.** The category of all fields, morphisms are field homomorphisms that take 1 to 1 and the composition is the usual composition of functions.

Example 1.6 ($\mathbf{Mod}_R, {}_R\mathbf{Mod}, \mathbf{Vect}_k$)**.** Objects are right (left) R - modules and as its morphisms are module homomorphisms. When $R = k$ is a field we have a category $\mathbf{Vect}_k$ of all vector spaces over k. When $R = \mathbb{Z}$ we have the category of all abelian groups, denoted by **Ab**.

Example 1.7 ($(X, \leq)$)**.** Given a partially ordered set $(X, \leq)$. Then there is a category X whose objects are elements of X and the hom set $\text{Hom}(x, y)$ between $x, y \in X$ is either empty or has only one elements; the second happens iff $x \leq y$. For instance, $X = \mathbf{n} = \{0, 1, ..., n-1\}$ with usual order is a category.

Example 1.8. Let (X, τ) be a topological space. We make τ a structure of paritially ordered sets under the ordinary inclusion. By example 1.6 this is a category and we denote by $O(X)$. If $U, V \in \tau$ and $U \subset V$ then the unique morphism from U to V is the inclusion which we denote by ι_U^V.

Example 1.9 (**Top**)**.** Objects are topological spaces, morphisms are continuous maps and composition is the usual composition of maps. There is also a category of pointed topological spaces $\mathbf{Top}_*$ whose objects are $(X, x_0), x \in X$ and morphism $f : (X, x_0) \to (Y, y_0)$ is a continuous one $f : X \to Y, f(x_0) = y_0$.

Example 1.10 (**HTop**)**.** Objects are all topological spaces, morphisms are homotopy classes of continuous maps. Composition is naturally defined but it is less obvious to see it is well-definied. Similar to $\mathbf{Top}_*$ there is a category $\mathbf{HTop}_*$.

Note. There are more interesting examples of a category as: algebra over a field, the category of graded rings (or graded algebra), the category of spectral sequences,... Some of them are useless, some are useful.

1.1 Special morphisms

From now on, whenever I write gf then it means f,g are composable and we deal with a fixed category $\mathcal{C}$. Capital letters like A is understood as an object of $\mathcal{C}$.

Definition 1.11 (**Monomorphism**)**.** A morphism $f : A \to B$ is called a **monomorphism** iff for every pair of morphism $x, y : C \to A$ then $fx = fy$ implies $x = y$.

Definition 1.12 (**Epimorphism**). A morphism $f : A \to B$ is called a **epimorphism** iff for every pair of morphisms $x, y : B \to C$ then $xf = yf$ imples $x = y$.

Definition 1.13 (**Isomorphism**). A morphism $f : A \to B$ is called a **isomorphism** if there exists a morphism $g : B \to A$ such that $gf = 1_A, fg = 1_B$. In this case, A and B are called isomorphic.

Note. Every isomorphism is both mono and epi but the converse is failed. Moreover, in some specific categories concepts of mono and epi are not compatible with the concreteness of injective, surjective.

1.2 Special Objects

Definition 1.14 (**Suboject**).

Definition 1.15 (**Quotient object**).

The next definition and examples are out of this section, we advice the reader to directly read the next section and comeback here lately.

Definition 1.16 (**(co)Group object**). Let $\mathcal{C}$ be a category having finite products and a terminal object Z. A ***group object*** in $\mathcal{C}$ is an object G and three morphisms $\mu : G \times G \to G, \eta : G \to G$ and $\epsilon : Z \to G$ making the followng diagrams commute:

- **Associvity.**

$$\begin{tikzcd} G \times G \times G \arrow[r, "1\times\mu"] \arrow[d, "\mu\times 1"] & G \times G \arrow[d, "\mu"] \\ G \times G \arrow[r, "\mu"] & G \end{tikzcd}$$

- **Identity.**

$$\begin{tikzcd} G \times Z \arrow[r, "1\times\epsilon"] \arrow[dr, "\lambda"'] & G \times G \arrow[d, "\mu"] & Z \times G \arrow[l, "\epsilon\times 1"'] \arrow[dl, "\rho"] \\ & G & \end{tikzcd}$$

  where λ, ρ are two projections

- **Inverse.**

$$\begin{tikzcd} G \arrow[r] \arrow[d, "\omega"] & G \times G \arrow[d, "\mu"] & G \arrow[l] \arrow[d, "\omega"] \\ Z \arrow[r, "\epsilon"] & G & Z \arrow[l, "\epsilon"'] \end{tikzcd}$$

  where ω is the only morphism from $G \to Z$ and two top horizotal arrows are $(1, \eta)$ and $(\eta, 1)$. A ***cogroup*** object in a category $\mathcal{D}$ having finite coproducts and an initial object is just a group object in its opposite category $\mathcal{D}^{op}$.

Example 1.17. A group object in **Grp** is just an abelian group.

Example 1.18. In **Sets** and **Top** the only group object is $\varnothing$. But in $\mathbf{Sets}_*$ and $\mathbf{Top}_*$ the only group object is $*$.

Example 1.19 (**Topological space**). A topological group is a group object in the category of topological spaces.

Example 1.20 (**Suspension and loop space**). In $\mathbf{HTop}_*$ the reduced suspension $\Sigma X = (X \times [0,1])/(X \times \{0,1\} \cup x_0 \times [0,1])$ is always a cogroup object while a loop space equipped compact-open topology $\Omega X = \mathrm{Hom}([0,1], X)$ is dually a group object. Note that both of them are self-functor $\mathbf{HTop}_* \to \mathbf{HTop}_*$.

Example 1.21. In algebraic geometry, a group object in the category of scheme is called a group scheme.

2 Functor

Once we have objects, categories we must pay attention to relations between them and that we have functors.

Before functoriality, people lived in caves. – B.Conrad

Definition 2.1 (Covariant functor)**.** Give two category $\mathcal{C}, \mathcal{D}$. A **covariant functor** $S : \mathcal{C} \to \mathcal{D}$ is a function such that:

(1) $S(C) \in \mathrm{Obj}(\mathcal{D})\ \forall C \in \mathrm{Obj}(\mathcal{C})$

(2) For each $f \in \mathrm{Hom}(C, C'), C, C \in \mathrm{Obj}(C)$ there is a morphism $S(f) : S(C) \to S(C')$

(3) $S(gf) = S(g)S(f)$ whenever there is a sequence $A \xrightarrow{f} B \xrightarrow{g} C$

(4) $S(1_A) = 1_{S(A)}\ \forall A \in \mathrm{Obj}(\mathcal{C})$

A **contravariant functor** $\mathcal{C} \to \mathcal{D}$ is just a functor $\mathcal{C}^{op} \to \mathcal{D}$.
Every algebraic invariance is constructed as a functor.

Definition 2.2 (**Faithful functor**)**.** A functor $S : \mathcal{C} \to \mathcal{D}$ is called faithful if $S : \mathrm{Hom}(A, B) \to \mathrm{Hom}(S(A), S(B))$ is injective for all $A, B \in \mathrm{Obj}(\mathcal{C})$.

Definition 2.3 (**Full functor**)**.** A functor $S : \mathcal{C} \to \mathcal{D}$ is called faithful if $S : \mathrm{Hom}(A, B) \to \mathrm{Hom}(S(A), S(B))$ is surjective for all $A, B \in \mathrm{Obj}(\mathcal{C})$.

Example 2.4 (**Dual vector space**)**.** Given a field k, then in the category of vector spaces over k there is a self-contravariant functor that sends each vector space V to its dual $V^* = \mathrm{Hom}_k(V, k)$. This point of view could apply to modules over ring rather than vector spaces.

Example 2.5 (**Forgetful functor**)**.** Let take a typical example. Then the functor **Grp** $\to$ **Sets** sending a group to its underlying set is called ***forgetful functor***. By mean of this, you can construct a number of forgetful functors.

Example 2.6 (**Free functor**)**.** Free functors are dual functors of forgetful functor, in most cases they are *adjoin*. Let take a typical examples, given a set S and a ring R. There is a free module, denoted by $F(S)$ whose elements are formal sum:

$$\sum_{s \in S, v_s \in R} v_s s$$

This is a free functor **Sets** $\to_R$ Mod.

Example 2.7 (**Grothendieck completion**)**.** Given an abelian monoid $(A, +)$ we define an equivalent relation on A^2 by $(a_1, a_2) \sim (b_1, b_2)$ if there exists a $c \in A : a_1 + b_2 + c = a_2 + b_1 + c$. Then the set $G(A) = A^2/\sim$ is really an abelian group. This construction can be viewed as a functor G : **AbMon** $\to$ **Ab**.

Example 2.8. Homotopy groups, singular homology groups are examples of convariant functor **HTop** $\to$ **Ab**(**Grp**).

Example 2.9. Singular cohomology groups or cohomology ring, topological K - theory are examples of contravariant functors

Example 2.10. Four most important functors in homological algebra, $\mathrm{Tor}, \mathrm{Hom}, \bigotimes, \mathrm{Ext}$ are all functorial.

Example 2.11. A **presheaf** on a topological space X is a convariant functor $O(X) \to$ **Sets**. Here **Sets** can be changed to **Ab** or $\mathbf{Mod}_R$. A **sheaf** is a presheaf but satisfies **equilazer conditions**, that is, **Uniqueness** and **Gluing**. There are two categories of presheaves and sheaves over X whose morphisms are natural transformations, they are denoted by $\mathbf{pSh}(X)$ and $\mathbf{Sh}(X)$, respectively.

Example 2.12 (**Spectrum**)**.** The correspondence $R \to \mathrm{Spec}(R)$ wherein the right side is spectra of a commutative ring R equipped Zariski topology, is a **contravariant** functor **CRing** $\to$ **Top**.

Example 2.13 (**Scheme**)**.** In algebraic geometry, a **scheme** is a convariant functor (the category of commutative rings) **CRings** $\to$ **Sets**. Roughly speaking, a system of polynomial under an action of a ring homomorphism is remained a system of polynomial.

Example 2.14 (**Direct and inverse image**)**.** See example 2.29.

Example 2.15. Given a topological space X, there is functor **Top** $\to$ **CRings** that maps a topological space to all continuous real-valued functions on it, $C(X) = C(X, \mathbb{R})$. A famous theorem due to both Gelfand and Kolmogorov asserts that in the category of compact Hausdorff spaces, $C(X)$ is isomorphic to $C(Y)$ as commutative rings implies that X is homeomorphic to Y. Because of this theorem, this functor is not useful as (co)homology functors since considering an isomorphism between two rings are as difficult as considering whether two spaces are homeomorphic.

2.1 Natural transformation

Definition 2.16 (**Natural transformation**)**.** Given two convariant functors (or both contravariant but arrows are reserved) $S, T : \mathcal{C} \to \mathcal{D}$. A **natural transformation** $\tau : S \to T$ is a one-parameter family of morphisms in $\mathcal{D}$ in each object of $\mathcal{C}$

$$\tau = (\tau_A : SA \to TA)_{A \in \mathrm{Obj}(\mathcal{C})}$$

making the following digram commutes for all morphism $f : A \to A'$

$$\begin{tikzcd} SA \arrow[r, "\tau_A"] \arrow[d, "Sf"] & TA \arrow[d, "Tf"] \\ SA' \arrow[r, "\tau_{A'}"] & TA' \end{tikzcd}$$

Some authors like the following representation:

$$\begin{tikzcd} S \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\tau"] & T \end{tikzcd}$$

If τ_A is a isomorphism for all $A \in \mathrm{Obj}(\mathcal{C})$ then we say S, T are **isomorphic as functors**.

Example 2.17 (**Characteristic classes**)**.** A characteristic class is a way of associating to each principal bundle a cohomology class. Formally, given a topological group G a **characteristic class** of X is a natural transformation $c : Prin_G(\_\_) \to H^*(\_\_)$.

- The first *Chern class* $c_1 : Prin_{U(1)}(X) \to H^2(X, \mathbb{Z})$ appears in complex geometry and is a complete invariant of line bundles.
- The first *Stiefel-Whitney class* $w_1 : Prin_{O(1)}(X) \to H^1(X, \mathbb{Z}/2)$ is completely similar, that is, w_1 is a isomorphism defined on all spaces.

There is a class (maybe proper class) of natural transformations between S, T and we denote it by $\mathrm{Nat}(S, T)$. Now there is a famous lemma which makes me feel uncomfortable in one year ago whenever I see it.

Lemma 2.18 (**Yoneda lemma**). *Let $\mathcal{C}$ be a category and $A \in Obj(\mathcal{C})$ and $S : \mathcal{C} \to$ **Sets** be a convariant functor. Then there is bijection:*

$$Nat(Hom_{\mathcal{C}}(A, \_\_), S) \to S(A)$$

given by $\tau \to \tau_A(1_A)$.

The prove is straightforward and can be found in any textbook on homological algebra.

Corollary 2.19. *Let $\mathcal{C}$ be a category and $A, B \in Obj(\mathcal{C})$ such that $Hom_{\mathcal{C}}(A, \_\_)$ is naturally isomorphic to $Hom_{\mathcal{C}}(B, \_\_)$ then $A \cong B$.*

Corollary 2.20 (**Yoneda embedding**). *Given $\mathcal{C}$ is a small category then there is a functor $S : \mathcal{C} \to$ $\textbf{Sets}^{\mathcal{C}^{op}}$ that is injective on objects and whose image is a fullsubcategory of $\textbf{Sets}^{\mathcal{C}^{op}}$. In other words, we say every small category is a full subcategory of a category of presheaves.*

Example 2.21 (**Double dual vector space**). This is a canonical example. In category of finite vector spaces over a field k then V is isomorphic to its dual V^* but this isomorphism is *unnatural* because it is depent on a choice of basises. Consider its double dual, again the finiteness of V implies V and V^{**} are isomorphic. But here one has a particular canonical isomorphism.

Example 2.22 (**Cohomology operation**). Cohomology operations become central to algebraic from about 20 years ago and has widely proved its power especially in homotopy. Given two abelian groups G, H then a **cohomology operation** of type (n, m, G, H) is a natural transformation between two cohomology theories $H^n(\_\_, G) \to H^m(\_\_, H)$ defined on **CW** or **Top** and may be imposed some extra conditions.

- The Steenrod squares $Sq^i : H^n(X, \mathbb{Z}/2) \to H^{n+i}(X, \mathbb{Z}/2)$.
- The Steenrod powers $P^i : H^n(X, \mathbb{Z}/p) \to H^{n+2i(p-1)}(X, \mathbb{Z})$ in which p is a fixed prime.
- The Adam operations $\psi^k : K(X) \to K(X)$ are used to solve the famous *Hopf invariant one* problem and *vector fields on spheres.*

Definition 2.23 (**Representable functor**). A functor is said to be **representable** if it is isomorphic to a either hom functor (convariance or contravariance), $\text{Hom}(\_\_, *)$ or $\text{Hom}(*, \_\_)$.

Example 2.24 (**Brown's representability**). For any abelian group G and any CW-complex X then by Brown's representability theorem the singular cohomology $H^n(X, n)$ is represented by $K(G, n)$ - an Eilenberg-Maclane space. It means there is natural bijection between $H^n(X, n)$ and all homotopy classes $[X, K(G, n)]$. Lostly speaking, the functor $H^n : \mathbf{CW} \to \mathbf{Ab}$ is representable.

We can said even more that a contravariant functor $h : (CW_0)^{pt} \to (\mathbf{Sets}^{pt})^{op}$ that send homotopy pushout to weak pullback and wedge sum in arbitrary of index set to product (strongly additive) (formularly, $h(\vee X_i) = \prod h(X_i)$) then h is representable.

Example 2.25 (**Algebrac K_0 group**). An unrelavant example but I also want to interpret here is the first algebraic K - group $K_0(R)$ of a ring R. It is Grothendieck completion of monoid of all finitely generate projective modules over R under the isomorphic relation. A well-known result claims that $K_0(R) \cong [\text{Spec}(R), \mathbb{Z}]$ wherein the right side is homotopy classes from spectra of R to $\mathbb{Z}$ with discrete topology.

Example 2.26 (**Topological K - theory**). The reduced complex topological K - theory $\widetilde{K}(X)$ is represented by classifying space BU of infinite unitary group U. It can be formulated as $\widetilde{K}(X) \cong [X, BU] \cong \langle X, BU \times \mathbb{Z} \rangle$. Real K - theory use BO instead of BU. Two significant notes here is the Bott periodicity theorem and Grassmanian manifolds.

- The Bott periodicity asserts that $\Omega^2 BU \cong BU \times \mathbb{Z}$ or equivalently $\widetilde{K}(X) \cong \widetilde{K}(S^2 X)$. ($S$ is stand for unreduced suspension). Note also that we have homotopy equivalences:

$$BU \times \mathbb{Z} \cong \Omega U \Rightarrow U \cong \Omega BU \cong \Omega(BU \times \mathbb{Z}) \cong \Omega^2 U$$

- The classifying space $BU(n)$ of n - dimensional unitary group $U(n)$ is homotopic to n - dimensional complex Grassmanian manifold $G_n = \mathrm{Gr}(n, \mathbb{C}^\infty)$ and the set of (unpointed) homotopy classes $[X, G_n]$ represents the set of isomorphism classes of n - dimensional complex vector bundles over X. In formula,
$$[X, G_n] \cong Vect^n(X)$$

Example 2.27 (**Classifying space of principal bundles**)**.** If we denote $Prin_G(X)$ for isomorphism classes of principal G - bundles over X then a theorem of Steenrod concludes that there is a space BG and a so-called G - bundle, *universal bundle* $EG \to BG$ which classifies all G - bundles over X. In formula:
$$[X, BG] \cong Prin_G(X)$$
When G is discrete we could take $K(G, 1)$ as BG and take its universal cover to be universal bundle. The problem arises when G is not discrete, like Lie groups then a famous construction due to Milnor is welcome to all of you.

Adjoint functors

We see that right (or left) adjoint functor of a functor (if exists) is unique up to an isomorphism and vice versa. Indeed, MacLane in his *Categories for Working Mathematician* proved that

Theorem 2.28. *If (S, T) and (S, T') are adjoint pairs where $S : \mathcal{C} \to \mathcal{D}, T, T' : \mathcal{D} \to \mathcal{C}$ then $T \cong T'$.*

Example 2.29 (**Adjoint to forgetful functor**)**.** Forgetful functors seems always have their adjoint functors. The following list proves my statement:

- For each set X we can form a free (abelian or not) group (or vector space) with a formal basis is this set. This construction give us two adjoints of forgetful from **Grps**, **Ab** to **Sets**.

- Given an arbitrary group H there are two ways to construct a new abelian which is closely related to G. That is, the center $c(G)$ and the abelianization $G/[G, G]$. But we do not prefer the first one since it is not a functorial way. The second one gives us an adjoint to forgetful functor **Ab** $\to$ **Grps**
$$\mathrm{Hom}_{\mathbf{Ab}}(G/[G, G], H) \cong \mathrm{Hom}_{\mathbf{Grps}}(G, H)$$

- Grothendieck completion is adjoint to forgetful funtor from the category of abelian groups to the category of abelian monoids.

- If $\mathcal{F}$ is a presheaf on a topological space X, then a morphism of presheaves $sh : \mathcal{F} \to \mathcal{F}^{sh}$ is a **sheafification** of $\mathcal{F}$ if $\mathcal{F}^{sh}$ is a sheaf and for any other sheaf $\mathcal{S}$ and any presheaf morphism $f : \mathcal{F} \to \mathcal{S}$ there *exists a uniques* morphism of sheaves $f^{sh} : \mathcal{F}^{sh} \to \mathcal{S}$ making the following diagram commutes:

$$\begin{tikzcd} \mathcal{F} \arrow[r, "sh"] \arrow[dr, "f"'] & \mathcal{F}^{sh} \arrow[d, dashed, "f^{sh}"] \\ & \mathcal{S} \end{tikzcd}$$

  The construction of sheafification is via the fact that the category of etale bundle over X and sheaves over X are isomorphic. An explicit construction ultilizes compatible germs is given in Ravi Vakil's book. Now it is easy to verify that sheafification is left adjoint to forgetful functor.

- Given two rings R, S and three modules $A_R, {}_R B_S, C_S$ then there is a natural isomorphism:

$$\mathrm{Hom}_S(A \otimes_R B, C) \to \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C))$$

  this example is written in every textbook of category theory or homological algebra as a first nontrivial adjoint pair.

- ring extension

- Given a field k and A is a k - module we form its ***tensor algebra***:

$$T(M) = \bigoplus_{n=0}^{\infty} A^{\otimes n}$$

  and tensor algebra of a k - module is left adjoint to forgetful functor $\mathbf{Alg}_k \to \mathbf{Mod}_k$:

$$\mathrm{Hom}_{\mathbf{Alg}_k}(T(A), B) \cong \mathrm{Hom}_{\mathbf{Mod}_k}(A, B)$$

  there are various examples like this, they appears in the theory of algebra over field and particularly in theory of Lie algebra.

Example 2.30 (**Digonal functor**)**.** The diagonal functor Δ has both left and right adjoints, that is limit and colimit. (See definition 3.14)

Example 2.31 (**Stalk and skyscraper sheaf**)**.** The stalk of a presheaf is given in example 3.5, it is a functor $\mathbf{Sh}(X) \to \mathbf{Ab}$ (in the case sheavese of abelian groups). If A is any abelian group we definie the ***skyscaper sheaf*** x_*A at the point $x \in X$ is $(x_*(A))(U) = A$ if $x \in U$ and 0 for otherwise. Therefore we have an adjoint pair:

$$\mathrm{Hom}_{\mathbf{Ab}}(\mathcal{F}_x, A) \cong \mathrm{Hom}_{\mathbf{Sh}(X)}(\mathcal{F}, x_*(A))$$

Example 2.32 (**Direct image and inverse image**)**.** Given a continuous map between two continuous map $f : X \to Y$ and a sheaf $\mathcal{G}$ on X we could define a sheaf on Y by $f_*\mathcal{G}(V) = \mathcal{G}(f^{-1}(V))$. It is called ***pushfoward*** by f and is adjoint to inverse image (example 3.6). That is,

$$\mathrm{Hom}_{\mathbf{Sh}(X)}(f^{-1}\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{\mathbf{Sh}(Y)}(\mathcal{F}, f_*(\mathcal{G}))$$

Example 2.33 (**Reduced suspension and hom**)**.** In $\mathbf{HTop}_*$, reduced suspension and loop is an adjoint pair.(example 1.22)

Example 2.34 (**Frobenius reciprocity**)**.** This is a theorem due to Frobenius which establishes the duality between the process of restricting and inducting. Let G be a group and H its a subgroup. Next, let Res^G_H denote class function of G to H and Ind^G_H the induced class function of a given class function on H. For any group K there is an inner product $\langle \_, \_ \rangle_K$ on the vector space of class function $K \to \mathbb{C}$. Then for any class function $\psi : H \to \mathbb{C}, \sigma : G \to \mathbb{C}$ there is a equality:

$$\left\langle \mathrm{Ind}^G_H \psi, \phi \right\rangle_G = \left\langle \psi, \mathrm{Res}^G_H \phi \right\rangle_H$$

in other words, they form an Hermitian adjoint pair.

Isomorphism and equivalence of categories

Definition 2.35. Given two categories $\mathcal{C}, \mathcal{D}$ we say they are **isomorphic** if there exists two functors $S : \mathcal{C} \to \mathcal{D}, T : \mathcal{D} \to \mathcal{C}$ such that $ST = id_{\mathcal{D}}, TS = id_{\mathcal{C}}$.

Example 2.36 (**Finite posets and finite T_0 - spaces**)**.** This outstanding simple discover brings a new way to look at finite spaces due to Alexandroff, for this reason posets are usually known as A - spaces. For our convinience, we restrict to finite poset and finite T_0 - spaces. The stragery here is when we have a finite space X then we assign each point x its minimal open set U_x which is defined to be intersection of all open sets containning x. After that, we definine a relation $x \leq y \Leftrightarrow U_x \subset U_y$. Conversely, if we have a poset X then an its subset U is open iff $x \in U \Rightarrow y \in Y \ \forall y \leq x$.

Example 2.37 (**Boolean algebras and Boolean rings**)**.** Boolean algebras and Boolean rings are isomorphic as categories. This is an exercise in chapter 1 of M.Atiyah, *Introduction to commutative algebra*. (In the case I remember rightly)

Example 2.38 (**Sheaves and etale spaces**)**.** An **etale space** is something like a *vector bundle of abelian groups*. Briefly, it is a local surjective homeomorphism in which every stalk is an abelian group with additions and inversions are continuous. An **etale map** is almost like a vector bundle morphism and there is a category of etale spaces of abelian groups $\mathbf{Sh}_{et}(X, \mathbf{Ab})$ over a topological space X. The sheaf of section of an etale space of abelian groups defines a functor $\mathbf{Sh}_{et}(X, \mathbf{Ab}) \to \mathbf{Sh}(X, \mathbf{Ab})$ and a intricated theorem claims that this is indeed an isomorphism of categories.

Example 2.39 (**Group rings and representations**)**.** Let G be a group and k is a field hence we have kG is the group algebra. Given a group representation $\rho : G \to GL(V)$ where V is a vector space over k we turn V into a kG - module by the following rule:

$$(\sum_{g \in G} v_g g)v = \sum_{g \in G} v_g \rho(g)(v)$$

Conversely, given a kG - module M then M is a k - vector space as well. A multiplication with an element of G yields a linear automorphism of M and hence we have a group homomorphism $\rho : G \to GL(M)$. This natural correspondence extends us an isomorphism of categories.

In fact, most of categories are not isomorphic and we want to weaken our condition but it should be still strong enough to reflect our desired properties.

Definition 2.40 (**Equivalence**)**.** Given two categories $\mathcal{C}, \mathcal{D}$ we say there are **equivalent** if there exists two fucntor $S : \mathcal{C} \to \mathcal{D}, T : \mathcal{D} \to \mathcal{C}$ such that $ST \cong id_{\mathcal{D}}, TS \cong id_{\mathcal{C}}$. (isomorphic as functors by mean of natural transformation)

Example 2.41 (**Algebraic bundles and topological bundles**)**.** The Serre-Swan theorem asserts that if X is a compact Hausdorff space and $C(X)$ is the ring of real-valued functions on X then the category real vector bundle over X is isomorphic to the category of finitely generated projective modules over $C(X)$. Particularly, algebraic line bundles are as equivalent as topological line bundles. (A **algebraic line bundle** over a ring R is a finitely generated projective module of constant rank 1 at every prime ideal of R)

Example 2.42 ($GL_n(\mathbb{R})$**-principal bundles and real vector bundles**)**.** Given a principal G - bundle and a linear representation $\rho : G \to \mathrm{Aut}(V)$, we get an associated vector bundle whose fibers look like V instead of G. This gives us an equivalence between the category of principal $GL_n(\mathbb{R})$ - bundles and the category of n - dimensional real vector bundles. The inverse functor is given by frame bundle.

There is another example called *Dold-Kan correspondence* which I give at the last big section.

2.2 Derive a new category from known categories

Definition 2.43. A **subcategory** $\mathcal{A}$ of a category $\mathcal{C}$ is a category whose objects of $\mathcal{A}$ are also objects of $\mathcal{C}$ and morphisms are morphisms in $\mathcal{C}$ ($\mathrm{Hom}_{\mathcal{A}}(A, B) \subset \mathrm{Hom}_{\mathcal{C}}(A, B)$).

Example 2.44. The category **Ab** of all abelian groups is a fullsubcategory of the category of all groups **Grp**.

Definition 2.45 (**Quotient category**)**.** A **congruence** on a category $\mathcal{C}$ is an equivalent relation $\sim$ on the class $\bigcup \mathrm{Hom}(A, B)$ of all morphisms in $\mathcal{C}$ such that:

- $f \in \mathrm{Hom}(A, B)$ and $f \sim f'$ then $f' \in \mathrm{Hom}(A, B)$.
- $f \sim f', g \sim g'$ and the composite gf exists imply that $gf \sim g'f'$.

Definition 2.46 (**Quotient category**)**.** A congruence $\sim$ on a category $\mathcal{C}$ defines a quotient category $\mathcal{C}/\sim$ whose objects are objects of $\mathcal{C}$ and morphisms are equivalent classes of morphisms in $\mathcal{C}$.

Example 2.47. The category **HTop** is a quotient category of the category **Top** under the homotopic relation.

Theorem 2.48. *Every functor $F : \mathcal{C} \to \mathcal{D}$ determines a congruence by saying $f \sim g$ iff $F(f) = F(g)$. Then this functor factors through the quotient functor $\mathcal{C} \to \mathcal{C}/\sim$.*

Definition 2.49 (**Product category**)**.** Given two categories $\mathcal{C}, \mathcal{D}$ there is a product category $\mathcal{C} \times \mathcal{D}$ whose objects

$$\mathrm{Obj}(\mathcal{C} \times \mathcal{D}) = \mathrm{Obj}(C) \times \mathrm{Obj}(D)$$

and morphisms

$$\mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((A, B), (A', B')) = \mathrm{Hom}_{\mathcal{C}}(A, A') \times \mathrm{Hom}_{\mathcal{D}}(B, B')$$

Definition 2.50 (**slice category**)**.** Given a category $\mathcal{C}$ and an object A. The **slice category** over A is written $(A \downarrow \mathcal{C})$ whose an its object is a pair (B, π_B) with $B \in \mathrm{Obj}(\mathcal{C}), \pi_B : B \to A$. A morphism from $(B, \pi_B) \to (B', \pi_{B'})$ is just a morphism in $\mathcal{C}$ from $B \to B$ making the followng diagram commutes:

$$\begin{tikzcd}
B \arrow[rr] \arrow[dr, "\pi_B"] & & B' \arrow[dl, "\pi_{B'}"] \\
& A &
\end{tikzcd}$$

Given a subcategory $\mathcal{C}'$ of $\mathcal{C}$ the category $(A \downarrow \mathcal{C}, \mathcal{C}')$ is just the category $(A \downarrow \mathcal{C})$ but morphisms are required to be morphisms in $\mathcal{C}'$.

Definition 2.51 (**General form of comma category**)**.** Given three categories $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and two functors $S : \mathcal{A} \to \mathcal{C}, T : \mathcal{B} \to \mathcal{C}$. There is a comma category $(S \downarrow T)$:

- Objects are triplet (A, B, h) wherein $h \in \mathrm{Hom}(S(A), T(B))$.
- Morphisms between $(A, B, h), (A', B', h')$ are pairs (f, g) with $f \in \mathrm{Hom}(A, A'), g \in \mathrm{Hom}(B, B')$ and the following square commutes:

$$\begin{tikzcd}
S(A) \arrow[r, "S(f)"] \arrow[d, "h"] & S(A') \arrow[d, "h'"] \\
T(B) \arrow[r, "T(g)"] & T(B')
\end{tikzcd}$$

Example 2.52. When $\mathcal{A} = \mathcal{C}$, S is identity functor and $\mathcal{B}$ is trivial category (with one object $*$ and one morphism) the category $(S \downarrow T)$ is called slice category over $T(*)$, and denoted by $(T(*) \downarrow \mathcal{A})$.

Example 2.53. When $\mathcal{A} = \mathcal{B} = \mathcal{C}$ and both S, T are identity functors then the comma category $(S \downarrow T) = \mathcal{C}^{\to}$ is called arrow category.

Example 2.54. There is a category of real (or, complex) vector bundles of topological spaces. Then this category has objects as a triplet (E, B, p) wherein $p : E \to B$ is a vector bundle. Given two vector bundles $(E_1, B_1, p_1), (E_2, B_2, p_2)$ then a morphism between them is a pair of continuous (f, g) making the following square commutes:

$$\begin{tikzcd} E_1 \arrow[r, "f"] \arrow[d, "p_1"] & E_2 \arrow[d, "p_2"] \\ B_1 \arrow[r, "g"] & B_2 \end{tikzcd}$$

and we requires f sends $p_1^{-1}(x)$ to $p_2^{-1}(g(x))$. This category is a subcategory of $\mathbf{Top}^{\rightarrow}$.

Example 2.55. Fixed a vector space X there is a category of vector bundles over X. It is a subcategory of $(X \downarrow \mathbf{Top})$. Morphisms between two bundles over X should be require more, it sends a fiber at a point to fiber at that point; just like example 2.6 when we take $B_1 = B_2 = X$ and $g = id_X$. Because of the importance of this category, we give it a notation $\mathbf{Vect}(X)$.

Definition 2.56 (**Functor category**)**.** Given two categories $\mathcal{C}, \mathcal{D}$ there is a **functor category** $\mathcal{D}^{\mathcal{C}}$ whose objects are functors $\mathcal{C} \to \mathcal{D}$ and morphisms are natural transformations. This seems a good definition except in general there are too much morphisms which means morphisms between two objects can be a proper class. To avoid this phenomenom, we require domain $\mathcal{C}$ to be a small category. In these cases we usually denote $\mathcal{I}$ rather than $\mathcal{C}$, the letter I refers to the word ***index***.

3 Limit and Colimit

We begin with familiar definitions of kernel, cokernel and see them as examples of pullback and pushout and even we have a more general setting, namely, limit and colimit.

Definition 3.1 (**zero object**)**.** A **zero object** 0 in a category $\mathcal{C}$ is an object with precisely one morphism in each hom set $\mathrm{Hom}(0, A), \mathrm{Hom}(A, 0)$.

If a category $\mathcal{C}$ has a zero object then this object is unique up to an isomorphism and we are able to define **zero map** $A \to B$ as factoring through zero object, $A \to 0 \to B$.

Definition 3.2 (**pullback**)**.** Given two morphism $f : B \to D, g : C \to D$. The pullback is an object A with two morphisms $\pi_1 : A \to B, \pi_2 : A \to C$ satisfying $f\pi_1 = g\pi_2$ and is universal with this property. It means if we have another triplet (P, p_1, p_2) with $fp_1 = gp_2$ then there is a unique morphism from $P \to A$ making the following diagram commutes:

$$\begin{tikzcd} P \arrow[dr, dotted] \arrow[drr, bend left, "p_1"] \arrow[ddr, bend right, "p_2"'] & & \\ & A \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & B \arrow[d, "f"] \\ & C \arrow[r, "g"] & D \end{tikzcd}$$

Once we defined pullback we define **pushout** of two morphisms $p : A \to B, q : A \to C$ to be the pullback of two morphisms $p^{op} : B \to A, q^{op} : C \to A$ in opposite category. This is a special case of limit in a category is dual to colimit in its opposite category. Finally, we define kernel of a morphism $f : A \to B$ to be the pullback of $(f : A \to B, 0 \to B)$ and cokernel to be pushout of $(f : A \to B, A \to 0)$.

As a typical candidate we give you the Seifert - Van Kampen theorem.

Example 3.3 (**Van Kampen theorem**)**.** The Van Kampen theorem asserts that for two given open path-connected subspace U, V of a topological space X such that U_1, U_2 is also path-connected then $\pi_1(X)$ is a pushout of the diagram $(i_1 : \pi_1(U_1 \cap U_2) \to \pi_1(U_1), i_2 : \pi_1(U_1 \cap U_2) \to \pi_1(U_2))$, here based point is in U_1, U_2. It is often drawed as follow:

$$\begin{tikzcd}
 & \pi_1(U_1) \arrow[dr] \arrow[drr, bend left, "j_1"] & & \\
\pi_1(U_1 \cap U_2) \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & \pi_1(U_1) *_{\pi_1(U_1 \cap U_2)} \pi_1(U_2) \arrow[r, dashed, "\cong"] & \pi_1(X) \\
 & \pi_1(U_2) \arrow[ur] \arrow[urr, bend right, "j_2"'] & &
\end{tikzcd}$$

We are not ready to define what **Image** is as kernel and cokernel but if it exists then it must naturally be kernel of a cokenerl.

Definition 3.4 (**Colimit, Limit**)**.** Let $\mathcal{I}$ be a small category, $\mathcal{C}$ be an arbitrary category, $\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{I}}$ be the diagonal functor. The **colimit** of a functor $S : \mathcal{I} \to \mathcal{C}$ is an object of $\mathcal{C}$, written $\mathrm{colim}_{i \in \mathcal{I}} S_i$ together with a natural transformation from S to $\Delta(\mathrm{colim}_{i \in \mathcal{I}} S_i)$, which is universal among natural transformation $S \to \Delta A, A \in \mathrm{Obj}(\mathcal{C})$. That means there exists an unique morphism $f : \mathrm{colim}_{i \in \mathcal{I}} S_i$ making the following diagram commutes:

$$\begin{tikzcd}
S \arrow[rr] \arrow[dr] & & \Delta\mathrm{colim}_{i \in \mathcal{I}} S_i \arrow[dl, "\Delta f"] \\
 & \Delta(A) &
\end{tikzcd}$$

The **limit** of a functor $S : \mathcal{I} \to \mathcal{C}$ is the colimit of $S^{op} : \mathcal{I}^{op} \to \mathcal{C}^{op}$.

As a first example, we have pullback and pushout are a colimit and a limit, respectively.
The importance of adjoint pair is expressed in the following theorerm whose proof is omitted.

Example 3.5 (**Stalks of a presheave**)**.** The concept of *stalk* is motivated by germs of holomorphic functions and germ of diferentiable functions. Given a presheaf $\mathcal{F}$ on a space X and $x \in X$ then the **stalk** of $\mathcal{F}$ at x is defined to be $\varinjlim_{x \in U} \mathcal{F}(U)$ where the index set runs over all neighborhoods of x.

Example 3.6 (**Inverse image**)**.** Given a continuous map between two topological spaces, $f : X \to Y$. Let $\mathcal{F}$ to be a sheaf on Y then we want to *pullback* $\mathcal{F}$ to X using f. Firstly, we define a presheaf $f^{-1}\mathcal{F}^{pre}(U) = \mathrm{colim}_{\pi(U) \in V} \mathcal{F}(V)$ and then take its sheafification $f^{-1}\mathcal{F} = (f^{-1}\mathcal{F}^{pre})^{sh}$.

Theorem 3.7. *Suppose $L : \mathcal{C} \to \mathcal{D}$ be left adjoint to $R : \mathcal{D} \to \mathcal{C}$, where $\mathcal{C}, \mathcal{D}$ are arbitrary categories. Then L preserves all colimits and R preserves all limits.*

Theorem 3.8 (**Presheaf as colimit**)**.** *Let $\mathcal{C}$ be a presheaf of sets (and possibly, abelian groups, modules, ...) then $\mathcal{C}$ can always be treated as a colimit in a slice category.*

Proof. Let $F \in \mathbf{Sets}^{\mathcal{C}^{op}}$ be a presheaf and we define a canonical representable convariant functor h_X by $h_X = \mathrm{Hom}_{\mathcal{C}}(\_\_, X)$. We form a slice category $(\Delta \downarrow X)$ in definition 2.52 by taking T to be a constant functor. Explicit, it is a category whose objects are morphisms $h_X \to F$ and morphisms $f : h_X \to h_Y$ is a morphism makig the followng diagram commutes:

$$\begin{tikzcd}
h_X \arrow[rr, "f"] \arrow[dr] & & h_Y \arrow[dl] \\
 & F &
\end{tikzcd}$$

and by Yoneda's lemma we can view f as induce by a morphism $X \to Y$. There is a functor:

$$\phi : (\mathcal{C} \downarrow X) \to \mathbf{Sets}^{\mathcal{C}^{op}}$$

$$(h_X \xrightarrow{f} F) \mapsto h_X$$

on the other hand, there is a map $\phi(f) \to F$, $\forall f \in (\Delta \downarrow X)$ that commutes with all diagram and hence they induce a map

$$\mathrm{colim}_{(\mathcal{C}\downarrow X)} \phi(f) \to F \ (1)$$

and we want to prove this is an isomorphism. By Yoneda's lemma for each $X \in \mathcal{C}, a \in F(X)$ there is a morphism $h_X \to F$ such that the identity of X is sent to a. It follows that (1) is surjective. To see its injectivity we show that each:

$$\mathrm{colim}\phi(f)(X) \to F(X)$$

is an injection . Suppose two elements $a_1 \in \phi(f_1)(X), a_2 \in \phi(f_2)(X)$ are mapped to a same element in $F(X)$. Then f_1, f_2 correspond to maps $h_{Y_1} \to F, h_{Y_2} \to F$ given by elements $\beta_1 \in F(Y_1), \beta_2 \in F(Y_2)$, and a_1, a_2 correspond to maps $g_1 : X \to Y_1, g_2 : X \to Y_2$. The fact they are the same thing in $F(X)$ means that pullback $g_1^*(\beta_1) = g_2^*(\beta)$. Let $\gamma = g_1^*(\beta_1) = g_2^*(\beta)$: We have a first diagram:

$$\begin{tikzcd}
h_X \arrow[rr, "f_1"] \arrow[dr, "\gamma"'] & & h_Y \arrow[dl, "\beta_1"] \\
& F &
\end{tikzcd}$$

and a similar second one:

$$\begin{tikzcd}
h_X \arrow[rr, "f_2"] \arrow[dr, "\gamma"'] & & h_Y \arrow[dl, "\beta_2"] \\
& F &
\end{tikzcd}$$

The first diagram shows that $a_1 \in h_{Y_1}(X)$ of the colimit is identified with the identity of $h_X(X)$ by f_1. Similar argument to a_2 we see that they are identical in $F(X)$ and this shows that (1) is injective.

In general, limit behaviors not as well as colimit so we restrict our index category to be directed. That is a poset such that for two arbitrary elements i, j there is a element k such that $i, j \leq k$. In this kind of category we can make a simpler description of the elements in limit and in some nice categories (Mod_R, for instance) limit preserves short exact sequences. But since we are in the field of category theory so if you want more homological algebra then I suggest [3] as a reference.

4 Abelian categories

Briefly speaking, abelian categories are where we can do homological algebra. Initially, abelian categories first appear in Tohoku paper of Grothendieck in which he wanted to unify two cohomology theories, sheaf cohomology and group cohomology, because these cohomology theories share many similar properties and these stable categories were developed as a language to research similarities.

Definition 4.1 (**Additive category**). A category $\mathcal{C}$ is **additive**

(1) $\mathrm{Hom}(A, B)$ is an abelian group for all $A, B \in \mathrm{Obj}(\mathcal{C})$.

(2) The following distributive laws hold with $X, Y \in \mathrm{Obj}(\mathcal{C})$:

$$b(f+g) = bf + bg, (f+g)a = fa + ga \ \forall a \in \mathrm{Hom}(X, A), f, g \in \mathrm{Hom}(X, Y), b \in \mathrm{Hom}(Y, B)$$

(3) $\mathcal{C}$ has a zero object.

(4) $\mathcal{C}$ is complete and cocomplete, that is to say $\mathcal{C}$ has finite products and finite coproducts.

Example 4.2. $\mathbf{Mod}_R, {}_R\mathbf{Mod}, \mathbf{pSh}(X, \mathbf{Ab}), \mathbf{Sh}(X, \mathrm{Ab})$ are typical examples for additive category but neither **CRings** or **Grp** is additive.

Definition 4.3 (**Additive functor**)**.** An **additive functor** T between two additive categories $\mathcal{C} \to \mathcal{D}$ is a functor such that $T : \mathrm{Hom}(A, B) \to \mathrm{Hom}(TA, TB), A, B \in \mathrm{Obj}(\mathcal{C})$ is a homomorphism of abelian groups.

Example 4.4. Hom functor of either variance and tensor product are examples of additive functors $\mathbf{Mod}_R \to \mathbf{Ab}$.

Definition 4.5 (**Abelian category**)**.** An **abelian category** is an additive category such that:

(1) Every morphism has a kernel and a cokernel.

(2) Every monomorphism is a kernel and every epimorphism is a cokernel.

Example 4.6. $\mathbf{Mod}_R, {}_R\mathbf{Mod}$ are abelian categories.

Example 4.7. This example is not obvious although it is the most important one. It is to say that $\mathbf{Sh}(X, \mathbf{Ab})$ is an abelian category.

Example 4.8. Let $\mathcal{A}$ be an abelian category and $\mathcal{I}$ be a small category then the functor category $\mathcal{A}^{\mathcal{I}}$ is an abelian category as well. As an corollary, we have $\mathbf{pSh}(X, \mathcal{A})$ is an abelian category.

Example 4.9. Let $\mathbf{Comp}(\mathcal{A})$ be the category of all complex chains in $\mathcal{A}$. This category has objects as chains of form:

$$(C_\bullet, d_\bullet) ... \overset{d_{n+2}}{\to} C_{n+1} \overset{d_{n+1}}{\to} C_n \overset{d_n}{\to} C_{n-1} \overset{d_{n-1}}{\to} ...$$

wherein $C_n \in \mathrm{Obj}(\mathcal{A}), d_n d_{n+1} = 0\ \forall n$. A morphism between $(C_\bullet, d_\bullet) \to (C'_\bullet, d'_\bullet)$ is a collection of morphisms (f_n) making every square commutes:

$$\begin{tikzcd}
... \arrow[r, "d_{n+1}"] & C_n \arrow[r, "d_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "f_{n-1}"] & ... \\
... \arrow[r, "d'_{n+1}"] & C'_n \arrow[r, "d'_n"] & C'_{n-1} \arrow[r, "d'_{n-1}"] & ...
\end{tikzcd}$$

If $\mathcal{A}$ is an abelian category then so is $\mathbf{Comp}(\mathcal{A})$.

Definition 4.10 (**Image of a morphism**)**.** Let $f : A \to B$ be a morphism in an abelian category then we define the **image** of f to be:

$$\mathrm{Im}(f) = \ker(\mathrm{coker}(f))$$

With definition of image in hands we define an sequence $A \overset{f}{\to} B \overset{g}{\to} C$ is **exact** iff $\mathrm{Im}(f) = \ker(g)$.

Definition 4.11 (**Exact category**)**.** A category $\mathcal{P}$ is an ***exact category*** if it is a fullsubcategory of an abelian category $\mathcal{A}$ and *closed under extension*, that is, if $0 \to A \to B \to C \to 0$ is an exact sequence in $\mathcal{A}$ with $A, C \in \mathrm{Obj}(\mathcal{P})$ then $B \in \mathrm{Obj}(\mathcal{P})$.

Exact categories come from algebraic K - theory. Grothendieck defined an abelian group $K_0(\mathcal{P})$ as an abelian group having generators $\mathbf{Obj}(\mathcal{P})$ and the relation $A + C = B$ if there is an exact sequence $0 \to A \to B \to C \to 0$. Later, in analogy with topological K - theory, Bass invented the ***Whitehead group*** $K_1(\mathcal{P})$. Finally, thanks to Quillen because he gave a general construction called ***Q* - construction** where he associated $\mathcal{P}$ with a new category $\mathbf{Q}\mathcal{P}$ and took its classifying space $B\mathbf{Q}\mathcal{P}$ and then defined:

$$K_i(\mathcal{P}) = \pi_i(B\mathbf{Q}\mathcal{P})$$

Of course, this general definition of Quillen agrees with earlier definitions of K_0, K_1 due to Grothendieck and Bass. To avoid abstract, one should just take the fullsubcategory of $\mathbf{Mod}_R$ contains all finitely generated projective modules over commutative ring R.

Definition 4.12 (**Projective and injective objects**)**.** An object P in an abelian category $\mathcal{A}$ is **projective** if for every epimorphism $g : B \to C$ and every $f : P \to C$, there exists $h : P \to B$ with $f = gh$.

$$\begin{tikzcd} & P \arrow[dl, dashed, "h"'] \arrow[d, "f"] \\ B \arrow[r, "g"'] & C \end{tikzcd}$$

An object E is **injective** in $\mathcal{A}$ if it is projective in $\mathcal{A}^{op}$.

Definition 4.13. An abelian category $\mathcal{A}$ has ***enough projectives*** if for every object A of $\mathcal{A}$ there exist an epimorphism $P \to A$ with P is projective. Dually, $\mathcal{A}$ has ***enough injectives*** if there is a monomorphism $A \to E$ in which E is injective.

An advantage of injective objects is that it allows us to define cohomology. Fortunately, most of our familiar abelian categories have enough injectives.

Theorem 4.14. *If $\mathcal{A}$ is an abelian category that is closed under products and has enough injective then **Sh**$(X, \mathcal{A})$ has enough injectives.*

How do we know our axioms of defining an abelian category is sufficient and enough? Two following theorems answer the question:

Theorem 4.15 (**Freyd-Heron-Lubkin**)**.** *If $\mathcal{A}$ is a small abelian category, then there is a convariant faithful exact functor $F : \mathcal{A} \to$ **Ab***

The second theorem due to Mitchell improves the one above.

Theorem 4.16 (**Mitchell**)**.** *If $\mathcal{A}$ is a small abelian category, then there is a convariant **full** faithful exact functor $F : \mathcal{A} \to$ **Ab***

The ***Meta theorem*** is omitted here, the readers who prefer could easily find it in any standard textbook on category theory.

4.1 Ext groups in an abelian category

If an abelian category $\mathcal{A}$ does not have enough injectives then we do not hope to define Tor functor but we could still define Ext functor of either variance. To do this, we remind that there is a $1-1$ correspondence between equivalent classes of extensions of A by B and $\mathrm{Ext}^1(A, B)$ where A, B are two modules over a commutative ring. Pullback the group structure on $\mathrm{Ext}^1(A, B)$ to the set of equivalent classes we have an addition on it which is well-known as ***Baer*** sum. By mimicking this process, we could recapture the group structure of $\mathrm{Ext}^1(A, B)$ in an arbitrary abelian category. This approach is due to Yoneda.

An element of $\mathrm{Ext}^n(A, B), A, B \in \mathrm{Obj}(\mathcal{A})$ is an equivalent class of exact sequence of form:

$$\xi : 0 \to B \to X_n \to ... \to X_1 \to A \to 0$$

The equivalence relation is generated by the relation $\xi \cong \xi'$ if there is a diagram:

$$\begin{tikzcd} \xi : & 0 \arrow[r] \arrow[d] & B \arrow[r] \arrow[d, equal] & X_n \arrow[r] \arrow[d] & ... \arrow[r] & X_1 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\ \xi' : & 0 \arrow[r] & B \arrow[r] & X'_n \arrow[r] & ... \arrow[r] & X'_1 \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

To add ξ and ξ' (now they may be in different classes) let $X_1^{''}$ be the pullback of X_1, X'_1 over A. Let $X_n^{''}$ be the pushout of X_n, X'_n under B and let Y_n be the quotient of $X_n^{''}$ by the skew diagonal copy of B. Then $\xi + \xi'$ is defined to be the class of:

$$0 \to B \to Y_n \to X_{n-1} \oplus X'_{n-1} \to ... \to X_2 \oplus X'_2 \to X_1^{''} \to A \to 0$$

5 Geometry of a category

5.1 Simplicial set

Simplicial set is a way we genelizes posets (as a category), directed graphs to higher dimensional objects. Since the decomposition of a CW - complex into cells allows us to compute homology of a such space more easily. Homotopy groups, however, remain difficult to compute and simplicial sets allow us to define homotopy groups in a combinatorial way through their relation to topological spaces. (note that this point of view does not help us compute homotopy groups more easily)

It was firstly introduced by Samuel Eilenberg in 1950. In this section, we give definitions and sum up basic results in the theory of simplicial objects. The most famous one may be the Dold-Kan theorem which established the correspondence between chain complex category and simplex category of an arbitrary abelian category $\mathcal{A}$.

Historically, at the beginning of category theory, it is just considered as a language to interpret and unify several branches of modern mathematics. Until Daniel Kan defined the so-called **Kan extension** it became truly a research field of higher mathematics. Kan extension is closely related to simplicial objects and we leave here a famous quotation:

Everything is just a Kan extension.

Let Δ be the ***simplex category*** whose objects are posets $\mathbf{n}, n \in \mathbb{N}$ and morphisms are convariant functors (equivalently, continuous functions preserve order).

Definition 5.1 (Simplicial object). A ***simplicial object*** X in a category $\mathcal{C}$ is a contravariant functor $X : \Delta \to \mathcal{C}$ or equivalently a convariant functor $X : \Delta^{op} \to \mathcal{C}$ and hence a presheaf. A ***simplicial set*** is just a simplicial object in **Sets**.

A morphism between two simplicial objects is just a natural transformation between them. Denote the category of simplicial objects in $\mathcal{C}$ by $\mathbf{Sim}\mathcal{C}$.

If we work out the above condition for a simplicial set then we have an alternative definition.

Definition 5.2 (Alternative definition). A simplicial set X is a sequence of sets (X_n) together with functions:

$$d_i : X_n \to X_{n-1}, i = \overline{0, n-1}$$

$$s_j : X_n \to X_{n+1}, j = \overline{0, n-1}$$

the ***face*** and ***degenacy*** map, respectively, which satisfy the **simplicial identities**:

$$d_i \circ s_j = \begin{cases} s_{j-1} \circ d_i, & \text{for } i < j \\ \text{identity}, & \text{for } i = j, j+1 \\ s_j \circ d_{i-1}, & \text{for } i > j+1 \end{cases}$$

$$d_i \circ d_j = d_{j-1} \circ d_i \text{ for } i < j$$

$$s_i \circ s_j = s_{j+1} \circ s_i \text{ for } i \leq j+1$$

Definition 5.3 (Standard n - simplex). The ***standard n - simplex*** is defined to be the functor $\Delta^n = \mathrm{Hom}_\Delta(\_\_, \mathbf{n})$.

Example 5.4 (Horn of simplex). The i^{th} **- horn** of Δ^n, written Λ^n_i, is the subsimplex containning the union of faces of Δ^n *except* the i^{th} one.

By theorem 5.2 with $\mathcal{C} = \Delta$ we see that:

$$X \cong \operatorname{colim}_{\Delta^n \to X} \Delta^n$$

where the colimit is taken in the slice category $(\Delta \downarrow X)$, that is, X is the colimit of a functor $S : (\Delta \downarrow X) \to \mathbf{Sets}^{\Delta^{op}}$ and

$$X_n = X(\mathbf{n}) \cong \operatorname{Nat}(\operatorname{Hom}_\Delta(\_\_, \mathbf{n}), X) = \operatorname{Hom}_{\mathbf{SimSets}}(\Delta^n, X)$$

A morphism in $\mathbf{Sim}\mathcal{C}$ can be described as a sequence of maps $f_n : X_n \to Y_n$ that commute with face and degenacy maps.

Definition 5.5 (**Nerve of a simplicial set**)**.** The ***nerve*** of a small category $\mathcal{C}$ is a simplicial set defined by the following data. Its n - simplicies $(N\mathcal{C})_n$ is the functor category $\mathcal{C}^{\mathbf{n}}$. Concretely, $(N\mathcal{C})_n$ can be thought as sequences of $n+1$ composable morphism in $\mathcal{C} : c_0 \to c_1 \to ... \to c_n$ and the face maps d_i drops the i^{th} - position from such sequences and the degenacy maps s_j lengthen the sequence by inserting an identity at position i.

Define the geometric realization of standard n - simplex to be

$$|\Delta^n| = \left\{(x_0, ...x_n) \in \mathbb{R}^n : x_i \in [0,1], \sum x_i = 1\right\}$$

Example 5.6. For each $n \geq 0$ there is a funtor $\operatorname{Sing}_n : \mathbf{Top} \to \text{Sets}$ sending X to continuous maps $|\Delta^n| \to X$ and these functors form a simplicial set, called singular simplicial set.

Definition 5.7 (**Geometric realiztion of a simplicial set**)**.** With the same notations as in the definition 5.2 we define ***geometric realization*** of X to be

$$|X| = \operatorname{colim}_{\Delta^n \to X} |\Delta^n|$$

that is, $|X|$ is the colimit of the functor $S : (\Delta \downarrow X) \to \mathbf{Top}$ where objects of $|\Delta|$ are $|\Delta^n|$ and morphisms are induced from morphisms in Δ.

Theorem 5.8. *The functor realization $|.|$ is left adjoint to the singular functor Sing. That is:*

$$|.| : \boldsymbol{Sets}^{\Delta^{op}} \leftrightarrow \boldsymbol{Top} : Sing$$

is an adjoint pair.

Proof.

$$\begin{aligned}
\operatorname{Hom}_{\mathbf{Top}}(|X|, Y) &\cong \operatorname{Hom}_{\mathbf{Top}}(\operatorname{colim}_{\Delta^n \to X} |\Delta^n|, Y) \\
&\cong \lim_{(\Delta \downarrow X)} \operatorname{Hom}_{\mathbf{Top}}(|\Delta^n|, Y) \\
&\cong \lim_{(\Delta \downarrow X)} \operatorname{Hom}_{\mathbf{Sets}}(\Delta^n, \operatorname{Sing}(Y)) \\
&\cong \operatorname{Hom}_{\mathbf{Sets}}(\operatorname{colim}_{(\Delta \downarrow X)} \Delta^n, Sing(Y)) \\
&\cong \operatorname{Hom}_{\mathbf{Sets}}(X, \operatorname{Sing}(Y))
\end{aligned}$$

□

The singular functor has another property, it is a *Kan complex.*

Definition 5.9 (**Kan extension**)**.** A simplicial set X is called a ***Kan complex*** if it is satisfied the *extension* condition, that is for every map $f : \Lambda_i^n \to X, 0 \leq i \leq n$ can be extended to a $\overline{f}$ making the following diagram commutes:

$$\begin{tikzcd}
\Lambda_i^n \arrow[r, "f"] \arrow[d, hook] & X \\
\Delta^n \arrow[ur, "\overline{f}"] &
\end{tikzcd}$$

Theorem 5.10. *The simplicial set Sing(X) is a Kan complex.*

Proof. For every inclusion $j_i : |\Lambda_i^n| \to |\Delta^n|$ admits a retraction $p : |\Delta^n| \to |\Lambda_i^n|, p \circ j_i = id$:

$$\begin{tikzcd} {|\Lambda_i^n|} \arrow[r, "j_i"] & {|\Delta^n|} \arrow[l, bend left, dashed, "p"] \end{tikzcd}$$

where p is given by projection parallel to the vector from the barycenter of i^{th} face of Δ^n to the i^{th} vertex of Δ^n. The extension now is equivalent to the extension of the following diagram:

$$\begin{tikzcd} {|\Lambda_i^n|} \arrow[r, "f'"] \arrow[d, hook] & {|X|} \\ {|\Delta^n|} \arrow[ur] & \end{tikzcd}$$

which can be solved by setting the dashed arrow by $f' \circ p$.

We define the so-called ***Kan fibration*** to finish this section:

Definition 5.11 (**Kan fibration**)**.** A morphism of simplicial set $f : X \to Y$ is called a ***Kan fibration*** if for every $0 \leq i \leq n$ and a commutative square:

$$\begin{tikzcd} \Lambda_i^n \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

there exists a dashed morphism making the diagram commutes.

5.2 Homotopy of simplicial set

Now we are able to define homology of simplicial sets. Firstly, let $F : \mathbf{Sets} \to \mathbf{Ab}$ be free standard functor sending a set to its free abelian with this set as a basis. This functor can be naturally extended to a functor, which we denote with the same notation $F : \mathrm{Sim}\mathcal{C} \to \mathrm{SimAb}$

Definition 5.12 (**Homology of simplicial set**)**.** With the symbols in definition 5.2 we denote:

$$C_n(X) = F(X_n), d = \sum_{i=0}^{n} (-1)^i F(d_i)$$

It is easy to see $d^2 = 0$ and hence we define the **homology** group of X is homology of the chain complex $(C_\bullet(X), d_\bullet)$.

Definition 5.13 (**Homotopy of simplicial set**)**.** Let $f, g : X \to Y$ be morphism of simplicial sets. A ***homotopy*** between f and g is a morphism

$$H : \Delta^1 \times X \to Y$$

such that $H_{|\{0\} \times X} = f, H_{|\{1\} \times Y} = g$

Let X be a Kan complex, then we define $\pi_0(K)$ to be equivalent classes of vertices of K (The imposed condition of X to be a Kan complex is just to ensure the simplicial homotopy is indeed an equivalent relation) and generally we could ultilize simplicial homotopy to create a model for simplicial homotopy groups $\pi_n(X)$ but we do not give those constructions explicitly here.

The things $\pi_\bullet$ behavioring exactly like usual homotopy groups and $\pi_\bullet$ form a nonabelian homological δ - functor. We do not pursue that complex things here and I just give a brief abstract definition of what an abstract homotopy group is.

Definition 5.14 (**Homotopy groups and normalized complex**)**.** For every simplicial object X in an abelian category $\mathcal{A}$ we associate it to the chain complex $\mathcal{N}(\mathcal{A})$ where:

$$\mathcal{N}_n(X) = \bigcap_{i=0}^{n} \ker(d_i : X_n \to X_{n-1})$$

and n^{th} differential $\partial_n = (-1)^n d_n$. The n^{th} - homology of this chain complex is called the n^{th} - homotopy group $\pi_n(X)$ of X.

The following theorem was discovered independently by both Kan and Doldd in 1957

Theorem 5.15 (**Dold-Kan correspondence**)**.** *For any abelian category $\mathcal{A}$, the normalized chain complex functor $\mathcal{N}$ is an equivalence of categories between* ***Sim***$\mathcal{A}$ *and* ***Comp***$\mathcal{A}$*.*

For more information, even the inverse functor of $\mathcal{N}$, we refer the reader to Charles Weibel, *An Introduction to homological algebra.*

5.3 Geometric realization of a small category

Let $\mathcal{C}$ be a small category then we could construct a CW - complex $B\mathcal{C}$ naturally out of C. This is closed related to classifying space of a disrete topological group G which can be considered as a small category with one object. $B\mathcal{C}$ is the geometric realization of the nerve $N\mathcal{C}$ and can be characterized in a simple way.

Definition 5.16 (**Geometric realization**)**.** The ***geometric realization*** of a small category $\mathcal{C}$ is the geometric realization of its nerve.

When we say a category $\mathcal{C}$ satisfies a geometrical property $\mathcal{P}$ we refer it to its geometric realization. For instance, say $\mathcal{C}$ is contractible meaning $B\mathcal{C}$ is contractible as topological space.

Example 5.17. Let $G = \mathbb{Z}/2$ is a group considered as a category with only one object then the geometric realization of G is infinite projective space $\mathbb{RP}^{\infty}$.

A natural transformation between two functors $\mathcal{C} \to \mathcal{D}$ is equivalent to a functor $\mathcal{C} \times \mathbf{2} \to \mathcal{D}$ and so we have:

Theorem 5.18. *A natural transformationn $\eta : S \to T$ between two functors $S, T : \mathcal{C} \to \mathcal{D}$ gives rise to homotopy $B\mathcal{C} \times [0,1] \to B\mathcal{D}$.*

Example 5.19. Any category $\mathcal{C}$ with an initial object is contractible since the functor $\mathcal{C} \to \mathbf{1}$ has left adjoint.

In definition 2.52 we set T to be the constant functor at a object A then we have a comma category, written $(S \downarrow A)$. Dually if S is constant functor at an object B we have dual comma category $(B \downarrow T)$. There is a couple of theorem due to Quillen:

Theorem 5.20 (**Theorem A of Quillen**)**.** *Let $S : \mathcal{C} \to \mathcal{D}$ be a functor such that $(A \downarrow S)$ is contractible for all $A \in Obj(A)$ then $BS : B\mathcal{C} \to B\mathcal{D}$ is a homotopy equivalence.*

Theorem 5.21 (**Theorem B of Quillen**)**.** *Let $S : \mathcal{C} \to \mathcal{D}$ be a functor such that for every morphism $A \to A'$ in $\mathcal{D}$, the induced functor $(A \downarrow S) \to (A' \downarrow S)$ is a homotopy equivalence then we have a homotopy fibration:*

$$B(A \downarrow S) \to B\mathcal{C} \to B\mathcal{D}$$

In particular, there is a long exact sequence of homotopy groups:

$$... \to \pi_{n+1}(B((A \downarrow S)) \to \pi_{n+1}(B\mathcal{C}) \to \pi_{n+1}(B\mathcal{D}) \to \pi_n(B((A \downarrow S)) \to ...$$

5.4 Localization and groupoid

Definition 5.22 (**Localization of a category**)**.** For each small category $\mathcal{C}$ and a set of morphism Mthen the ***localization*** of $\mathcal{C}$ at M is a pair consists of a category $\mathcal{C}[M^{-1}]$ and a functor $q : \mathcal{C} \to \mathcal{C}[M^{-1}]$ satisfies the following properties:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "q"] \arrow[dr, dashed, "S"'] & \mathcal{C}[M^{-1}] \arrow[d, "S_M"] \\ & \mathcal{D} \end{tikzcd}$$

- For every $m \in M$ we have $q(m)$ is an isomorphism.
- For each functor $S : \mathcal{C} \to \mathcal{D}$ such that $F(m)$ is a isomorphism for all $m \in M$ then there exists an unique lifting functor $S_M : \mathcal{C}[M^{-1}] \to \mathcal{D}$ and a natural isomorphism $S_M \circ q \cong S$.
- The composition functor between two following functor categories is fully faithful:

$$(-) \circ q : \mathcal{D}^{\mathcal{C}[M^{-1}]} \to \mathcal{D}^{\mathcal{C}}$$

The universal property above shows that if localization exists then it is unique up to an isomorphism. In the case M is the set of all morphisms in $\mathcal{C}$ we write $\mathcal{C}[M^{-1}] = L\mathcal{C}$ and call it the global localization of $\mathcal{C}$.

Definition 5.23 (**Groupoid**)**.** A ***groupoid*** is a category in that every morphism is an isomorphism.

Note.

- Groupoid, localization and Quillen's theorem have various applications in the topology of finite spaces.
- Localization always exists and can be constructed explicitly. It appears in model category.

Example 5.24. Every global localization is a groupoid.

Example 5.25. Every group is a groupoid with only one object and arrows are indexed by its elements.

Example 5.26 (**Fundamental groupoid**)**.** We ascociate each topological space X with the so-called thing, the fundamental groupoid ΠX whose objects are points in X and:

$$\mathrm{Hom}(x, y) = \{[p] : \text{p is a path in X from x to y}\}$$

where the relation here is path-homotopy. The fundamental group of X with base point x_0 is then:

$$\pi_1(X, x_0) = \mathrm{Aut}_{\Pi X}(x_0)$$

Definition 5.27 (**Morphism of groupoids**)**.** A ***morphism of two groupoids*** is just a functor between them.

By mimicking all the constuction in classical group theory we define the following objects. From now on, we say an element of a groupoid to mean a morphism in it.

Definition 5.28 (**Kernel and image**)**.** Given a morphism $S : \mathcal{C} \to \mathcal{D}$ between two groupoids then its ***kernel*** is set of elements a in $\mathcal{C}$ such that $S(a)$ is an identity in $\mathcal{D}$ and its ***image*** is set of element of form $S(a)$ with a is an element in $\mathcal{C}$.

Definition 5.29 (**Normal subgroupoid**)**.** Given a groupoid $\mathcal{C}$ and a subgroupoid $\mathcal{N}$ (consider as a subcategory) is said to be ***normal*** if $\mathrm{Obj}(\mathcal{C}) = \mathrm{Obj}(\mathcal{N})$ and $\mathcal{N}$ is kernal of a morphism with domain $\mathcal{C}$.

Definition 5.30 (Quotient groupoid). Given a groupoid $\mathcal{C}$ and a normal subgroupoid $\mathcal{N}$ we define a relation in sets of morphisms and objects of $\mathcal{C}$. Two objects $x \sim y$ iff $\mathrm{Hom}_{\mathcal{N}}(x, y) \neq \varnothing$ and two elements g, h are equivalent iff there exists two elements a, b of $\mathcal{N}$ such that $agb = h$. The coset $\mathcal{C}/\mathcal{N}$ then is a groupoid, called quotient groupoid. It is characterized by the follwing universal property:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "p"] \arrow[dr, dashed, "S"'] & \mathcal{C}/\mathcal{N} \arrow[d, "\overline{S}"] \\ & \mathcal{D} \end{tikzcd}$$

that is, give a morphism $S : \mathcal{C} \to \mathcal{D}$ such that $S(m)$ is identity in $\mathcal{D}$ for all elements m of $\mathcal{N}$ then there is uniquely a morphism $\overline{S}$ factoring S through the canonical projection $p : \mathcal{C} \to \mathcal{C}/\mathcal{N}$.