FOUR APPLICATIONS OF RCF AND LCF THEOREMS

Vasile Cîrtoaje

Abstract

In this paper are presented four new and difficult symmetric inequalities with right convex and left concave functions, as applications of RCF-Theorem and LCF-Theorem from [1] and [2]. Note that all the functions involved in the proposed inequalities have more than one inflexion point.

1. Introduction

In [1] and [2], we have proved the following theorems:

Right Convex Function Theorem (RCF-Theorem). *Let* $f(u)$ *be a function defined on an interval* $\mathbf{I}\subset\mathbf{R}$ *and convex for* $u\ge s$, $s\in\mathbf{I}$. *If the inequality*

$$f(x_1)+f(x_2)+\cdots+f(x_n)\ge nf\left(\frac{x_1+x_2+\cdots+x_n}{n}\right)$$

holds for all $x_1,x_2,\cdots,x_n\in\mathbf{I}$ *such that*

$$x_2=x_3=\cdots=x_n\ge s \quad and \quad \frac{x_1+x_2+\cdots+x_n}{n}=s,$$

then it also holds for all $x_1,x_2,\cdots,x_n\in\mathbf{I}$ *such that* $\frac{x_1+x_2+\cdots+x_n}{n}\ge s$.

Left Concave Function Theorem (LCF-Theorem). *Let* $f(u)$ *be a function defined on an interval* $\mathbf{I}\subset\mathbf{R}$ *and concave for* $u\le s$, $s\in\mathbf{I}$. *If the inequality*

$$f(x_1)+f(x_2)+\cdots+f(x_n)\le nf\left(\frac{x_1+x_2+\cdots+x_n}{n}\right)$$

holds for all $x_1,x_2,\cdots,x_n\in\mathbf{I}$ *such that*

$$x_1=x_2=\cdots=x_{n-1}\le s \quad and \quad \frac{x_1+x_2+\cdots+x_n}{n}=s,$$

then it also holds for all $x_1,x_2,\cdots,x_n\in\mathbf{I}$ *such that* $\frac{x_1+x_2+\cdots+x_n}{n}\le s$.

Remark 1.1. The hypothesis in RCF-Theorem is equivalent to the condition that $f(x)+(n-1)f(y)\ge nf(s)$ for all $x,y\in\mathbf{I}$ such that $x\le s\le y$ and $x+(n-1)y=ns$.

Remark 1.2. Let $g(u)=\frac{f(u)-f(s)}{u-s}$. The hypothesis in RCF-Theorem is equivalent to the condition that $g(x)\le g(y)$ for all $x,y\in\mathbf{I}$ such that $x\le s\le y$ and $x+(n-1)y=ns$.

Remark 1.3. The hypothesis in LCF-Theorem is equivalent to the condition that $(n-1)f(x)+f(y)\le nf(s)$ for all $x,y\in\mathbf{I}$ such that $x\le s\le y$ and $(n-1)x+y=ns$.

Remark 1.4. Let $g(u)=\frac{f(u)-f(s)}{u-s}$. The hypothesis in LCF-Theorem is equivalent to the condition that $g(x)\ge g(y)$ for all $x,y\in\mathbf{I}$ such that $x\le s\le y$ and $(n-1)x+y=ns$.

In this paper, following closely theorems above, we will prove the following four statements.

Proposition 1.1. *If* $a_1,a_2,\cdots,a_n$ *are nonnegative real numbers such that*

$$a_1+a_2+\cdots+a_n=n,$$

then

$$\frac{a_1^2+a_2^2+\cdots+a_n^2-n}{(a_1^2-a_1)^2+(a_2^2-a_2)^2+\cdots+(a_n^2-a_n)^2}\le n-2+\frac{1}{n-1}\ . \tag{1}$$

Proposition 1.2. *If* a,b,c *are positive real numbers such that* $abc=1$, *then*

$$a^2+b^2+c^2-3\ge 18(a+b+c-ab-bc-ca)\,. \tag{2}$$

Proposition 1.3. *If* $a_1,a_2,\cdots,a_8$ *are nonnegative real numbers such that* $a_1a_2\cdots a_8\le 1$, *then*

$$\frac{1-a_1}{(1+a_1)^2}+\frac{1-a_2}{(1+a_2)^2}+\cdots+\frac{1-a_8}{(1+a_8)^2}\ge 0\,. \tag{3}$$

Proposition 1.4. *If* a_1,a_2,a_3,a_4,a_5 *are positive real numbers such that* $a_1a_2a_3a_4a_5\ge 1$, *then*

$$\frac{1+a_1}{1+a_1^2}+\frac{1+a_2}{1+a_2^2}+\frac{1+a_3}{1+a_3^2}+\frac{1+a_4}{1+a_4^2}+\frac{1+a_5}{1+a_5^2}\le 5\,. \tag{4}$$

2. Proofs of the proposed inequalities

Proof of Proposition 1.1. Let $A = n-2+\frac{1}{n-1}$, $A \ge 1$. Since $a_1+a_2+\cdots+a_n = n$, we may write (1) as

$$f(a_1)+f(a_2)+\cdots+f(a_n) \ge nf(\frac{a_1+a_2+\cdots+a_n}{n}), \tag{5}$$

where $f(u) = A(u^2-u)^2-u^2+1$, $u \ge 0$. The second derivative,

$$f''(u) = 12A(u^2-u)+2(A-1),$$

shows that $f(u)$ has two inflexion points for $u \ge 0$. Since $f''(u) \ge 0$ for $u \ge 1$, the function f is right convex for $u \ge s$, where

$$s = \frac{a_1+a_2+\cdots+a_n}{n} = 1 \ .$$

By RCF-Theorem, it suffices to prove (5) for $a_1 \le 1 \le a_2 = a_3 = \cdots = a_n$ and $a_1+a_2+\cdots+a_n = n$. According to Remark 1.2, this means to show that $g(x) \le g(y)$ for $0 \le x \le 1 \le y$ and $x+(n-1)y = n$, where

$$g(u) = \frac{f(u)-f(1)}{u-1} = A(u^3-u^2)-u-1.$$

We have

$$g(x)-g(y) = (x-y)[A(x^2+xy+y^2)-A(x+y)-1] =$$

$$= n(n-1)(1-y)(Ay-n+1)^2 \le 0,$$

and the proof is complete. Equality occurs only if $a_1 = \frac{1}{n^2-3n+3}$ and $a_2 = a_3 = \cdots = a_n = 1+\frac{n-2}{n^2-3n+3}$, or any cyclic permutation thereof.

♦

Proof of Proposition 1.2. We will show that for any real numbers x, y, z with $\frac{x+y+z}{3} = 0$, the inequality holds

$$f(x)+f(y)+f(z) \le 3f(\frac{x+y+z}{3}), \tag{6}$$

where $f(u) = 18(e^u - e^{-u}) - e^{2u}$, $u \in \mathbf{R}$. Replacing then x, y, z by $\ln a, \ln b, \ln c$, respectively, the desired inequality (2) follows.

In order to prove (6), we will apply LCF-Theorem to the function f defined on **R**, with $s = 0$. From the second derivative

$$f''(u)=18(e^{u}-e^{-u})-4e^{2u},$$

it follows that $f(u)$ has two inflexion points in **R**. Since $f''(u)<0$ for $u\le 0$, the function $f(u)$ is left concave for $u\le s=0$. According to LCF-Theorem, it suffices to consider only the case $x=y\le 0$. This means to prove the initial inequality for $a=b\le 1$ and $a^2c=1$. Then, the inequality successively becomes

$$2a^2+c^2-3\ge 18(2a+c-a^2-2ac),$$

$$2a^6-3a^4+1+18a^2(a^4-2a^3+2a-1)\ge 0,$$

$$(a^2-1)^2(2a^2+1)+18a^2(a-1)^3(a+1)\ge 0,$$

$$(a-1)^2(2a-1)^2(a+1)(5a+1)\ge 0.$$

The last inequality is clearly true, and the proof is completed. Equality occurs when $(a,b,c)=(1,1,1)$, and also when $(a,b,c)=(\frac{1}{2},\frac{1}{2},4)$ or any cyclic permutation thereof.

♦

Proof of Proposition 1.3. According to Lemma 2.1 and Lemma 2.2 below, it suffices to consider the case where all $a_i\le 3$ and $a_1a_2\cdots a_8=1$. We will show that for all $x_i\le \ln 3$ such that $x_1+x_2+\cdots+x_8=0$, the inequality holds

$$f(x_1)+f(x_2)+\cdots+f(x_8)\le 5f\left(\frac{x_1+x_2+\cdots+x_8}{8}\right), \tag{7}$$

where $f(u)=\dfrac{1-e^{u}}{(1+e^{u})^2}$. Replacing then each x_i by $\ln a_i$, the required inequality (3) follows.

We will prove (7) by applying RCF-Theorem to the function f defined on $\mathbf{I}=(-\infty,\ln 3]$, with $s=0$. Taking derivatives, we get

$$f''(u)=\frac{e^{u}(8e^{u}-e^{2u}-3)}{(1+e^{u})^4},$$

which shows that f has two inflexion point in **R**. We first have to show that $f(u)$ is convex for $s\le u\le \ln 3$; this means that $f''(u)\ge 0$ for $0\le u\le \ln 3$ or, equivalently, $8t-t^2-3\ge 0$ for $1\le t\le 3$. This is true since

$$8t-t^2-3\ge 8t-3t-3=5t-3>0.$$

According to RCF-Theorem, it suffices to prove the inequality (7) for $0\le x_2=x_3=\cdots=x_8\le \ln 3$ and $x_1+x_2+\cdots+x_8=0$; that is, to prove the initial inequality (3) for $1 \le a_2 = a_3 = \cdots = a_8 \le 3$ and $a_1 a_2 \cdots a_8 = 1$. Thus, we must show that

$$\frac{1-a}{(1+a)^2} + \frac{7(1-b)}{(1+b)^2} \ge 0$$

for $1 \le b \le 3$ and $ab^7 = 1$. Taking into account that

$$\frac{1-a}{(1+a)^2} = \frac{b^7(b^7-1)}{(b^7+1)^2},$$

we have to show that

$$\frac{b^7(b^6+b^5+b^4+b^3+b^2+b+1)}{(b^6-b^5+b^4-b^3+b^2-b+1)^2} - 7 \ge 0.$$

Since

$$b^6-b^5+b^4-b^3+b^2-b+1 = b^4(b^2-b+1)-(b-1)(b^2+1) \le b^4(b^2-b+1),$$

it suffices to prove the inequality

$$\frac{b^6+b^5+b^4+b^3+b^2+b+1}{b(b^2-b+1)^2} - 7 \ge 0.$$

This inequality is equivalent to $(b-1)^6 \ge 0$, which is clearly true. Equality in the given inequality occurs if and only if $a_1 = a_2 = \cdots = a_8 = 1$.

Lemma 2.1. *If the inequality* (3) *holds for any* $0 < a_i \le 3$ *such that* $a_1 a_2 \cdots a_8 = 1$, *then it holds for any* $0 \le a_i \le 3$ *such that* $a_1 a_2 \cdots a_8 \le 1$.

Proof. Assume that $0 \le a_i \le 3$ and $a_1 a_2 \cdots a_8 \le 1$. Always there are eight positive numbers b_i such that $b_1 b_2 \cdots b_8 = 1$ and $a_i \le b_i \le 3$ for $i = 1, 2, \cdots, 8$. According to the hypothesis, the inequality holds $\sum_{i=1}^{8} \frac{1-b_i}{(1+b_i)^2} \ge 0$. Since the function $f(x) = \frac{1-x}{(1+x)^2}$ has the derivative $f'(x) = \frac{x-3}{(1+x)^3} < 0$ for $x \in [0,3)$, $f(x)$ is strictly decreasing on $[0,3]$. Therefore, $\frac{1-a_i}{(1+a_i)^2} \ge \frac{1-b_i}{(1+b_i)^2}$ for all subscripts i, and hence

$$\sum_{i=1}^{8} \frac{1-a_i}{(1+a_i)^2} \ge \sum_{i=1}^{8} \frac{1-b_i}{(1+b_i)^2} \ge 0.$$

Lemma 2.2. *If the inequality* (3) *holds for any* $0 \le a_i \le 3$ *such that* $a_1 a_2 \cdots a_8 \le 1$, *then it holds for any* $a_i \ge 0$ *such that* $a_1 a_2 \cdots a_8 \le 1$.

Proof. Assume that all $a_i \ge 0$, and $a_1 a_2 \cdots a_8 \le 1$. Define the numbers $x_1, x_2, \cdots, x_8$ as

$$x_i = \begin{cases} a_i & , \text{ for } a_i \le 3 \\ \dfrac{a_i + 3}{a_i - 1} & , \text{ for } a_i > 3 \end{cases}.$$

It is easy to show that $0 \le x_i \le 3$, $x_i \le a_i$ and $\dfrac{1 - x_i}{(1 + x_i)^2} = \dfrac{1 - a_i}{(1 + a_i)^2}$ for all i. Since $0 \le x_i \le 3$ and $x_1 x_2 \cdots x_8 \le a_1 a_2 \cdots a_8 \le 1$, from the hypothesis we have $\sum_{i=1}^{8} \dfrac{1 - x_i}{(1 + x_i)^2} \ge 0$, and hence $\sum_{i=1}^{8} \dfrac{1 - a_i}{(1 + a_i)^2} \ge 0$.

Remark 2.1. If $n = 9$, then the inequality $\sum_{i=1}^{n} \dfrac{1 - a_i}{(1 + a_i)^2} \ge 0$ is not true for any positive numbers a_i with $\prod_{i=1}^{n} a_i = 1$. Indeed, for $a_2 = a_3 = \cdots = a_9 = 3$, the inequality becomes $\dfrac{1 - a_1}{(1 + a_1)^2} - 1 \ge 0$, which is false.

♦

Proof of Proposition 1.4. According to Lemma 2.3 and Lemma 2.4 below (which can be proved in the same way as the preceding Lemma 2.1 and Lemma 2.2), it suffices to consider the case where all $a_i \ge \sqrt{2} - 1$ and $a_1 a_2 a_3 a_4 a_5 = 1$. In this case, the inequality can be proved by applying LCF-Theorem to the function $f(u) = \dfrac{1 + e^u}{1 + e^{2u}}$ defined on $\mathbf{I} = [\ln(\sqrt{2} - 1), \infty)$, with $s = 0$. The second derivative

$$f''(u) = \frac{e^u (1 - 4e^u - 6e^{2u} + 4e^{3u} + e^{4u})}{(1 + e^{2u})^3},$$

shows that f has four inflexion point in **R**. Finally, we have to prove the inequality for $\sqrt{2} - 1 \le a_1 = a_2 = a_3 = a_4 \le 1$ and $a_1 a_2 a_3 a_4 a_5 = 1$; that is

$$\frac{4(1 + a)}{1 + a^2} + \frac{1 + b}{1 + b^2} \le 5$$

for $\sqrt{2} - 1 \le a \le 1$ and $a^4 b = 1$. Since

$$\frac{1+b}{1+b^2}=\frac{a^4(1+a^4)}{1+a^8} \quad \text{and} \quad \frac{1+a^4}{1+a^8}\le\frac{2}{1+a^4}\le\frac{4}{(1+a^2)^2},$$

we get

$$5-\frac{4(1+a)}{1+a^2}-\frac{1+b}{1+b^2}\ge 5-\frac{4(1+a)}{1+a^2}-\frac{4a^4}{(1+a^2)^2}=$$

$$=\frac{1-4a+6a^2-4a^3+a^4}{(1+a^2)(1+a^4)}=\frac{(1-a)^4}{(1+a^2)(1+a^4)}\ge 0,$$

which completes the proof. Equality holds only if $a_1=a_2=a_3=a_4=a_5=1$.

Lemma 2.3. *If the inequality* (4) *holds for any* $a_i\ge\sqrt{2}-1$ *such that* $a_1a_2a_3a_4a_5=1$, *then it holds for any* $a_i\ge\sqrt{2}-1$ *such that* $a_1a_2a_3a_4a_5\ge 1$.

Lemma 2.4. *If the inequality* (4) *holds for any* $a_i\ge\sqrt{2}-1$ *such that* $a_1a_2a_3a_4a_5\ge 1$, *then it holds for any* $a_i>0$ *such that* $a_1a_2a_3a_4a_5\ge 1$.

Remark 2.2. If $n=6$, then the inequality $\sum_{i=1}^{n}\frac{1+a_i}{1+a_i^2}\le 6$ is not true for any positive numbers a_i with $\prod_{i=1}^{n}a_i=1$. Indeed, for $a_2=a_3=a_4=a_5=a_6=\frac{1}{2}$, the inequality becomes $\frac{1+a_1}{1+a_1^2}\le 0$, which is false.

References

[1] V. Cîrtoaje, A generalization of Jensen's Inequality, *Gazeta Matematica – Seria A*, 2 (2005), 124-138.

[2] V. Cîrtoaje, *Algebraic Inequalities. Old and New Method*, Gil Publishing House, 2006.

[3] M. Tetiva, A new proof for the Right Convex Function Theorem, *Gazeta Matematica – Seria A*, 2 (2006), 126-133.

Vasile Cîrtoaje

Department of Automatics and Computers

Petroleum-Gas University

Ploiesti City, *Bucuresti* 39, *Romania*

email: vcirtoaje@upg-ploiesti.ro