CANADIAN MATHEMATICAL OLYMPIAD 2011
PROBLEMS AND SOLUTIONS

(1) Consider 70-digit numbers n, with the property that each of the digits $1, 2, 3, \ldots, 7$ appears in the decimal expansion of n ten times (and 8, 9, and 0 do not appear). Show that no number of this form can divide another number of this form.

Solution. Assume the contrary: there exist a and b of the prescribed form, such that $b \geq a$ and a divides b. Then a divides $b - a$.

Claim: a is not divisible by 3 but $b - a$ is divisible by 9. Indeed, the sum of the digits is $10(1 + \cdots + 7) = 280$, for both a and b. [Here one needs to know or prove that an integer n is equivalent of the sum of its digits modulo 3 and modulo 9.]

We conclude that $b - a$ is divisible by $9a$. But this is impossible, since $9a$ has 71 digits and b has only 70 digits, so $9a > b > b - a$. $\square$

(2) Let $ABCD$ be a cyclic quadrilateral whose opposite sides are not parallel, X the intersection of AB and CD, and Y the intersection of AD and BC. Let the angle bisector of $\angle AXD$ intersect AD, BC at E, F respectively and let the angle bisector of $\angle AYB$ intersect AB, CD at G, H respectively. Prove that $EGFH$ is a parallelogram.

Solution. Since $ABCD$ is cyclic, $\Delta XAC \sim \Delta XDB$ and $\Delta YAC \sim \Delta YBD$. Therefore,

$$\frac{XA}{XD} = \frac{XC}{XB} = \frac{AC}{DB} = \frac{YA}{YB} = \frac{YC}{YD}.$$

Let s be this ratio. Therefore, by the angle bisector theorem,

$$\frac{AE}{ED} = \frac{XA}{XD} = \frac{XC}{XB} = \frac{CF}{FB} = s,$$

and

$$\frac{AG}{GB} = \frac{YA}{YB} = \frac{YC}{YD} = \frac{CH}{HD} = s.$$

Hence, $\frac{AG}{GB} = \frac{CF}{FB}$ and $\frac{AE}{ED} = \frac{DH}{HC}$. Therefore, $EH||AC||GF$ and $EG||DB||HF$. Hence, $EGFH$ is a parallelogram. $\square$

(3) Amy has divided a square up into finitely many white and red rectangles, each with sides parallel to the sides of the square. Within each white rectangle, she writes down its width divided by its height. Within each red rectangle, she writes down its height divided by its width. Finally, she calculates x, the sum of these numbers. If the total area of the white rectangles equals the total area of the red rectangles, what is the smallest possible value of x?

Solution. Let a_i and b_i denote the width and height of each white rectangle, and let c_i and d_i denote the width and height of each red rectangle. Also, let L denote the side length of the original square.

Lemma: Either $\sum a_i \geq L$ or $\sum d_i \geq L$.

Proof of lemma: Suppose there exists a horizontal line across the square that is covered entirely with white rectangles. Then, the total width of these rectangles is at least L, and the claim is proven. Otherwise, there is a red rectangle intersecting every horizontal line, and hence the total height of these rectangles is at least L. □

Now, let us assume without loss of generality that $\sum a_i \geq L$. By the Cauchy-Schwarz inequality,

$$\begin{aligned} \left(\sum \frac{a_i}{b_i}\right) \cdot \left(\sum a_i b_i\right) &\geq \left(\sum a_i\right)^2 \\ &\geq L^2. \end{aligned}$$

But we know $\sum a_i b_i = \frac{L^2}{2}$, so it follows that $\sum \frac{a_i}{b_i} \geq 2$. Furthermore, each $c_i \leq L$, so

$$\sum \frac{d_i}{c_i} \geq \frac{1}{L^2} \cdot \sum c_i d_i = \frac{1}{2}.$$

Therefore, x is at least 2.5. Conversely, $x = 2.5$ can be achieved by making the top half of the square one colour, and the bottom half the other colour. □

(4) Show that there exists a positive integer N such that for all integers $a > N$, there exists a contiguous substring of the decimal expansion of a that is divisible by 2011. (For instance, if $a = 153204$, then 15, 532, and 0 are all contiguous substrings of a. Note that 0 is divisible by 2011.)

Solution. We claim that if the decimal expansion of a has at least 2012 digits, then a contains the required substring. Let the decimal expansion of a be $a_k a_{k-1} \ldots a_0$. For $i = 0, \ldots, 2011$, Let b_i be the number with decimal expansion $a_i a_{i-1} \ldots a_0$. Then by pidgenhole principle, $b_i \equiv b_j \mod 2011$ for some $i < j \leq 2011$. It follows that 2011 divides $b_j - b_i = c \cdot 10^i$. Here c is the substring $a_j \ldots a_{i+1}$. Since 2011 and 10 are relatively prime, it follows that 2011 divides c. □

(5) Let d be a positive integer. Show that for every integer S, there exists an integer $n > 0$ and a sequence $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$, where for any k, $\epsilon_k = 1$ or $\epsilon_k = -1$, such that

$$S = \epsilon_1(1+d)^2 + \epsilon_2(1+2d)^2 + \epsilon_3(1+3d)^2 + \cdots + \epsilon_n(1+nd)^2.$$

Solution. Let $U_k = (1+kd)^2$. We calculate $U_{k+3} - U_{k+2} - U_{k+1} + U_k$. This turns out to be $4d^2$, a constant. Changing signs, we obtain the sum $-4d^2$.

Thus if we have found an expression for a certain number S_0 as a sum of the desired type, we can obtain an expression of the desired type for $S_0 + (4d^2)q$, for any integer q.

It remains to show that for any S, there exists an integer S' such that $S' \equiv S \pmod{4d^2}$ and S' can be expressed in the desired form. Look at the sum

$$(1+d)^2 + (1+2d)^2 + \cdots + (1+Nd)^2,$$

where N is "large." We can at will choose N so that the sum is odd, or so that the sum is even.

By changing the sign in front of $(1+kd)^2$ to a minus sign, we decrease the sum by $2(1+kd)^2$. In particular, if $k \equiv 0 \pmod{2d}$, we decrease the sum by 2 (modulo $4d^2$). So

If N is large enough, there are many $k < N$ such that k is a multiple of $2d$. By switching the sign in front of r of these, we change ("downward") the congruence class modulo $4d^2$ by $2r$. By choosing N so that the original sum is odd, and choosing suitable $r < 2d^2$, we can obtain numbers congruent to all odd numbers modulo $4d^2$. By choosing N so that the original sum is even, we can obtain numbers congruent to all even numbers modulo $4d^2$. This completes the proof. $\square$