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A Baire Category Approach to the Existence of Solutions of Multivalued Differential Equations in Banach Spaces

By

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§ 1. Introduction.

Let X be a real Banach space. Let $\mathscr{B}$ be the set of all closed convex bounded subsets of X which have nonempty interior. In this note we study the solution sets of the multivalued differential equations

$$\dot{x} \in F(t, x) \quad x(t_0)=x_0 \quad \left(\cdot=\frac{d}{dt}\right), \tag{1.1}$$

$$\dot{x} \in \partial F(t, x) \quad x(t_0)=x_0. \tag{1.2}$$

Here, F is a mapping from an open subset of $\boldsymbol{R}\times X$ into $\mathscr{B}$ and $\partial F(t, x)$ denotes the boundary of $F(t, x)$.

Our main result states that, if X is reflexive and F continuous in the Hausdorff metric, then (1.2) has at least one solution. We obtain this as an immediate consequence of a more general theorem which establishes that almost all (in the sense of the Baire category) solutions of (1.1) are actually solutions of (1.2).

If X has finite dimension, our existence result is a special case of Filippov's theorem [5]; but it is new when X is infinite dimensional. In this case most existence theorems refer to equation (1.1) and are obtained under compactness assumptions on F ([1], [3]). Further properties of multivalued differential equations (1.1) with nonconvex $F(t, x)\subset\boldsymbol{R}^n$ can be found in [7].

We adapt here a method used by Cellina [2] in the study of a differential inclusion in $\boldsymbol{R}$.

Denote by $\mathscr{M}_F$ (resp. $\mathscr{M}_{\partial F}$) the set of all solutions of (1.1) (resp. (1.2)). We shall prove, first of all, that $\mathscr{M}_F$ is nonempty and that, under the metric of uniform convergence, is a complete metric space. We show, next, that $\mathscr{M}_{\partial F}$ can be expressed as a countable intersection of open dense subsets of $\mathscr{M}_F$. Thus $\mathscr{M}_{\partial F}$ is a dense G_δ-subset of $\mathscr{M}_F$, hence it is nonempty and (1.2) has solutions.

When F is single valued, (1.1) and (1.2) reduce to the same ordinary differential equation which, as is well known, has not necessarily solutions if F is only continuous and X is an infinite dimensional space ([6], [9]). This shows that for continuous F, in infinite dimensional Banach spaces, the existence of solutions of (1.2) can fail, without the assumption that $F(t, x)$ have nonempty interior.

§ 2. Notations and main results.

Let X be a real Banach space with norm $|\cdot|$. In any Banach space we denote by $S(u, r)$ the open ball with centre at u and radius $r>0$. We put $S=S(0, 1)\subset X$. For any set $A\subset X$, ∂A stands for the boundary of A.

Denote by $\mathscr{K}$ (resp. $\mathscr{B}$) the space of all nonempty subsets of X which are bounded (resp. closed convex bounded with nonempty interior). $\mathscr{K}$ is endowed with the Hausdorff pseudometric

$$h(A, B)=\inf\{t>0 \mid A\subset B+tS,\ B\subset A+tS\}, \qquad A, B\in \mathscr{K}.$$

As well known, h becomes a metric when is restricted to $\mathscr{B}$. For any $x\subset X$ and $A \subset X$, $A\neq\phi$, we set $d(x, A)=\inf\{|x-a| \mid a\in A\}$. In the space $\boldsymbol{R}\times X$ we use the norm $|(t, x)|=\max\{|t|, |x|\}$, $(t, x)\in \boldsymbol{R}\times X$.

Let F be a continuous mapping from a nonempty open subset of $\boldsymbol{R}\times X$ into $\mathscr{B}$. Let (t_0, x_0) be in the domain of F. We wish to prove the existence of (local) solutions of (1.1) and (1.2). To this end, if we consider the restriction of F to a nonempty open subset of its domain, say $\Omega_2=J_{2a}\times D_{2R}$, where $J_{2a}=(t_0-2a, t_0+2a)$ and $D_{2R}=S(x_0, 2R)$, we can assume without loss of generality, that:

(*) F is a continuous mapping from Ω_2 into $\mathscr{B}$ and satisfies $h(F(t, x), 0)<M$ for each $(t, x)\in\Omega_2$.

By a *solution* of (1.1) (resp. (1.2)) we mean any function $x\colon [t_0, T]\to X$ $(t_0<T)$ which is Lipschitzean, has derivative a.e. and satisfies (1.1) (resp. (1.2)) for almost all $t\in[t_0, T]$.

Proposition 2.1. *Let F satisfy* (*). *Then* (1.1) *has at least one solution x: $[t_0, T] \to X$, where $0<T-t_0<\min\{a, R/M\}$. Moreover, if X is reflexive, the uniform limit of solutions is a solution of* (1.1).

Denote by $\mathscr{M}_F$ (resp. $\mathscr{M}_{\partial F}$) the set of all solutions of (1.1) (resp. (1.2)) which are defined on $[t_0, T]$. If X is reflexive, by Proposition 2.1, $\mathscr{M}_F$ is a nonempty closed subset of the Banach space $C([t_0, T], X)$. Consequently, $\mathscr{M}_F$ is a nonempty complete metric space under the metric induced by the norm of uniform convergence of $C([t_0, T], X)$.

Our purpose is to show that $\mathscr{M}_{\partial F}$ is a dense G_δ-subset of $\mathscr{M}_F$. To this end, for any $\theta>0$, we set

$$\mathscr{N}_\theta=\left\{x\in\mathscr{M}_F \,\middle|\, \int_{t_0}^{T} d(\dot{x}(s), \partial F(s, x(s)))ds<\theta\right\}.$$

Since $(t, x) \to \partial F(t, x)$ is a continuous mapping from Ω_2 into $\mathscr{K}$ and $\dot{x}$ is measurable, the function under the integral is measurable.

Proposition 2.2. *Let F satisfy $(*)$. Then, for any $\theta > 0$, the set $\mathscr{N}_\theta$ is dense in $\mathscr{M}_F$. If, in addition, X is reflexive, the set $\mathscr{N}_\theta$ is open in $\mathscr{M}_F$.*

By virtue of Propositions 2.1 and 2.2 we obtain immediately the following

Theorem 2.3. *Let X be a reflexive real Banach space. Let F satisfy $(*)$. Then the set $\mathscr{M}_{\partial F}$ is a dense G_δ-subset of $\mathscr{M}_F$ and hence, in particular, $\mathscr{M}_{\partial F}$ is nonempty.*

Proof. Let $\theta_1 > \theta_2 > \cdots$ be such that $\theta_n \to 0$ as $n \to +\infty$. By Proposition 2.1 $\mathscr{M}_F$ is a complete metric space and, by Proposition 2.2, the sets $\mathscr{N}_{\theta_n}$ are open and dense in $\mathscr{M}_F$. Therefore

$$\mathscr{N} = \bigcap_{n=1}^{\infty} \mathscr{N}_{\theta_n}$$

is a dense G_δ-subset of $\mathscr{M}_F$ and so $\mathscr{N}$ is nonempty. Since $\mathscr{N} = \mathscr{M}_{\partial F}$, the theorem is proved.

§ 3. Proof of Proposition 2.1.

Suppose that F satisfies $(*)$. Let us introduce the following function $\sigma: \Omega_2 \to \boldsymbol{R}$ defined by

$$\sigma(t, x) = \tfrac{1}{2} \sup \{r > 0 \mid \text{there is } y \in F(t, x) \text{ such that } S(y, r) \subset F(t, x)\}.$$

Since F is continuous and takes values in $\mathscr{B}$, it follows that σ is continuous and positive [4, Lemma 3.1].

We denote by $L^p([t_0, T], X)$, $1 \le p < +\infty$, the Banach space of all (strongly measurable functions $u: [t_0, T] \to X$ such that

$$\int_{t_0}^{T} |u(t)|^p \, dt < +\infty,$$

equipped with norm

$$\left(\int_{t_0}^{T} |u(t)|^p \, dt \right)^{1/p}.$$

We set $\Omega_1 = J_a \times D_R$ where, $J_a = (t_0 - a, t_0 + a)$ and $D_R = S(x_0, R)$.

Proof of Proposition 2.1. Let $v_0 \in F(t_0, x_0)$ be such that $d(v_0, \partial F(t_0, x_0)) > \sigma(t_0, x_0)$. Let $t_1 = \sup \{t_0 \le \tau \le T \mid d(v_0, \partial F(t, x_0 + (t - t_0)v_0)) > 0,$ for each $t \in [t_0, \tau]\}$. Define $x_1: [t_0, t_1] \to X$ by

$$x_1(t) = x_0 + (t - t_0)v_0, \qquad t \in [t_0, t_1].$$

Note that $(t, x_1(t)) \in \Omega_1$ for $t \in [t_0, t_1]$. Now, suppose that $x_n\colon [t_{n-1}, t_n] \to X$, $n \geq 1$, has been defined and satisfies $(t, x_n(t)) \in \Omega_1$ for each $t \in [t_{n-1}, t_n]$. Let $v_n \in F(t_n, x_n(t_n))$ be such that $d(v_n, \partial F(t_n, x_n(t_n))) > \sigma(t_n, x_n(t_n))$. Let $t_{n+1} = \sup\{t_n \leq \tau \leq T \mid d(v_n, \partial F(t, x_n(t_n) + (t - t_n)v_n)) > 0$ for each $t \in [t_n, \tau]\}$. Then, define $x_{n+1}\colon [t_n, t_{n+1}] \to X$ by

$$x_{n+1}(t) = x_n(t_n) + (t - t_n)v_n, \qquad t \in [t_n, t_{n+1}].$$

Clearly $(t, x_{n+1}(t)) \in \Omega_1$ if $t \in [t_n, t_{n+1}]$. Thus the sequence of functions $\{x_n\}$ is well defined. Denote by x the piecewise linear function which is equal to x_n on $[t_{n-1}, t_n]$, $n = 1, 2, \cdots$. Observe that, by construction, $t_1 \leq t_2 \leq \cdots$; moreover, $t_n < t_{n+1}$ whenever $t_n < T$.

We claim that, for some n, $t_n = T$. Suppose the contrary. Then $\{t_n\}$ is strictly increasing and, since it is bounded, it has a limit $\hat{t}$, $\hat{t} \leq T$. Set $\hat{x} = x(\hat{t})$ and fix $0 < \varepsilon < \sigma(\hat{t}, \hat{x})$. By the continuity of F and σ, there is $\delta > 0$ such that $|t - \hat{t}| < \delta/[2(M+1)]$ and $|x - \hat{x}| < \delta$ imply

$$h(\partial F(t, x), \partial F(\hat{t}, \hat{x})) < \frac{\varepsilon}{4}, \qquad |\sigma(t, x) - \sigma(\hat{t}, \hat{x})| < \frac{\varepsilon}{4}.$$

Fix n such that $|t_n - \hat{t}| < \delta/[2(M+1)]$ and $|x_n(t_n) - \hat{x}| < \delta/2$. For each $t \in [t_n, \hat{t}]$ we have $|x_n(t_n) + (t - t_n)v_n - \hat{x}| < \delta$, thus

$$\begin{aligned} d(v_n, \partial F(t, x_n(t_n) + (t - t_n)v_n)) &\geq d(v_n, \partial F(t_n, x_n(t_n))) \\ &\quad - h(\partial F(t_n, x_n(t_n)), \partial F(\hat{t}, \hat{x})) - h(\partial F(\hat{t}, \hat{x}), \partial F(t, x_n(t_n) + (t - t_n)v_n)) \\ &> \sigma(t_n, x_n(t_n)) - \frac{\varepsilon}{4} - \frac{\varepsilon}{4} > \sigma(\hat{t}, \hat{x}) - \frac{\varepsilon}{4} - \frac{\varepsilon}{4} - \frac{\varepsilon}{4} > 0. \end{aligned}$$

Hence $t_{n+1} \geq \hat{t}$, which is a contradiction. Thus there is n such that $t_n = T$ and, clearly, $x\colon [t_0, T] \to X$ is a solution of (1.1).

Assume, now, X reflexive and let $\{z_n\}$ be a sequence of solutions of (1.1) converging uniformly to z. We want to prove that z is a solution of (1.1). In fact $\{\dot{z}_n\}$, as a bounded set contained in the reflexive Banach space $L^2([t_0, T], X)$ (see [8], p. 89), is weakly precompact. By Eberlein-Smulian's theorem a subsequence, say $\{\dot{z}_n\}$, converges weakly to a measurable function $\omega \in L^2([t_0, T], X)$; hence, by Mazur's theorem ([8], p. 36, Corollary) a sequence of convex combinations $\{\sum_{i=0}^{k_n} \alpha_i^n \dot{z}_{n+i}\}$ converges strongly to ω in $L^2([t_0, T], X)$ and so also in $L^1([t_0, T], X)$. As a consequence of this, from

$$\sum_{i=1}^{k_n} \alpha_i^n z_{n+i}(t) = x_0 + \int_{t_0}^{t} \left(\sum_{i=0}^{k_n} \alpha_i^n \dot{z}_{n+i}(s) \right) ds, \qquad t \in [t_0, T],$$

letting $n \to +\infty$, it follows

$$z(t)=x_0+\int_{t_0}^{t}\omega(s)ds, \qquad t\in[t_0, T].$$

Since F is continuous and takes closed convex values, by a standard argument one shows that the Lipschitzean function z is a solution of (1.1). This completes the proof.

§ 4. Proof of Proposition 2.2 ($\mathcal{N}_\theta$ is dense).

In this section we prove the first statement of Proposition 2.2 namely, that the set $\mathcal{N}_\theta$ is dense in $\mathcal{M}_F$.

Let F satisfy $(*)$. For $\mu>0$ and $(t, x)\in\Omega_2$, set

$$F_\mu(t, x)=\{u\in F(t, x)\,|\,d(u, \partial F(t, x))\leq\mu\},$$
$$\Phi_\mu(t, x)=\{u\in F(t, x)\,|\,d(u, \partial F(t, x))>\mu\},$$
$$G_\mu(t, x)=\{u\in F(t, x)\,|\,d(u, \partial F(t, x))=\mu\}.$$

Let $(\hat{t}, \hat{x})\in\Omega_2$ and let $\theta<\mu<\sigma(\hat{t}, \hat{x})$. Then $F_\mu(\hat{t}, \hat{x})$ and $G_\mu(\hat{t}, \hat{x})$ are in $\mathcal{K}$ and, $\Phi_\mu(\hat{t}, \hat{x})$ is a nonempty convex open bounded subset of X [4, Remark 3.3]. Furthermore, it follows from [4, Remark 3.8] that there is a neighborhood V of $(\hat{t}, \hat{x})$ such that the mappings $(t, x)\to\Phi_\mu(t, x)$ and $(t, x)\to G_\mu(t, x)$ (respectively, from V to the nonempty convex open bounded subsets of X and, from V to $\mathcal{K}$) are well defined and continuous in V.

For any $x\in\mathcal{M}_F$ and $\mu>0$, put

$$\Delta_x^\mu=\{t\in[t_0, T]\,|\,d(\dot{x}(t), \partial F(t, x(t)))>\mu\}.$$

Lemma 4.1. *Let F satisfy $(*)$. Let $x\in\mathcal{M}_F$ and fix $\varepsilon>0$. Let*

$$0<\mu<\min\{\sigma(t, x(t))\,|\,t\in[t_0, T]\}$$

and suppose that Δ_x^μ has Lebesgue measure $m(\Delta_x^\mu)>0$. Let $t_0<\tau<T$ be a point of density of Δ_x^μ. Then, there exists $\lambda_0(\tau)>0$ such that for each $0<\lambda<\lambda_0(\tau)$ there is a function $z_{\tau,\lambda}$: $J_{\tau,\lambda}\to X$, $J_{\tau,\lambda}=[\tau-\lambda, \tau+\lambda]\subset[t_0, T]$, which is Lipschitzean, differentiable a.e. and such that

(4.1) $$z_{\tau,\lambda}(\tau\pm\lambda)=x(\tau\pm\lambda),$$

(4.2) $$|z_{\tau,\lambda}(t)-x(t)|<\varepsilon \qquad \textit{for each } t\in J_{\tau,\lambda},$$

(4.3) $$\dot{z}_{\tau,\lambda}(t)\in F_\mu(t, z_{\tau,\lambda}(t)) \qquad \textit{a.e. in } J_{\tau,\lambda}.$$

Proof. Let $\varepsilon>0$. Let τ be a point of density of Δ_x^μ. From this and the continuity of $G_{\mu/2}$ and Φ_μ at $(\tau, x(\tau))$ it follows that there is $0<\delta(\tau)<\min\{\varepsilon, a, R\}$ such that:

$$\frac{m(J_{\tau,\lambda}\backslash\Delta_x^\mu)}{m(J_{\tau,\lambda})}<\frac{\mu}{8M} \qquad \text{for each } 0<\lambda\leq\delta(\tau);$$

moreover, for each $(t, y) \in S((\tau, x(\tau)), \delta(\tau))$, the sets $G_{\mu/2}(t, y)$, $\Phi_\mu(t, y)$ are respectively nonempty bounded, nonempty convex open bounded and satisfy

$$G_{\mu/2}(\tau, x(\tau)) \subset G_{\mu/2}(t, y) + \frac{\mu}{2} S \subset F_\mu(t, y), \tag{4.4}$$

$$\Phi_\mu(t, y) \subset \Phi_\mu(\tau, x(\tau)) + \frac{\mu}{8} S.$$

Fix $0 < \lambda_0(\tau) < \delta(\tau)/[8(M+1)]$. Note that for each $s \in J_{\tau, \lambda_0(\tau)}$ we have $(s, x(s)) \in S((\tau, x(\tau)), \delta(\tau)/2)$. Let $J_{\tau,\lambda}$ be any closed interval $[\tau-\lambda, \tau+\lambda]$, $0<\lambda<\lambda_0(\tau)$. We have

$$q = \int_{J_{\tau,\lambda}} \dot{x}(s)\, ds = \int_{J_{\tau,\lambda} \cap \Delta_x^\mu} \dot{x}(s)\, ds + \int_{J_{\tau,\lambda} \setminus \Delta_x^\mu} \dot{x}(s) ds.$$

For each $s \in J_{\tau,\lambda} \cap \Delta_x^\mu$, we have $\dot{x}(s) \in \Phi_\mu(s, x(s)) \subset \Phi_\mu(\tau, x(\tau)) + (\mu/8)S$, thus

$$\begin{aligned}\int_{J_{\tau,\lambda} \cap \Delta_x^\mu} \dot{x}(s) ds &\in m(J_{\tau,\lambda} \cap \Delta_x^\mu)\left[\Phi_\mu(\tau, x(\tau)) + \frac{\mu}{8} S\right] \\ &\subset m(J_{\tau,\lambda})\left[\Phi_\mu(\tau, x(\tau)) + \frac{\mu}{8} S\right] + m(J_{\tau,\lambda} \setminus \Delta_x^\mu) MS \\ &\subset m(J_{\tau,\lambda})\left[\Phi_\mu(\tau, x(\tau)) + \frac{\mu}{4} S\right].\end{aligned}$$

On the other hand,

$$\left| \int_{J_{\tau,\lambda} \setminus \Delta_x^\mu} \dot{x}(s)\, ds \right| \le Mm(J_{\tau,\lambda} \setminus \Delta_x^\mu) < \frac{\mu}{8} m(J_{\tau,\lambda}).$$

Therefore

$$q \in m(J_{\tau,\lambda})[\Phi_\mu(\tau, x(\tau)) + \tfrac{3}{8}\mu S]$$

and so,

$$\frac{q}{m(J_{\tau,\lambda})} \in \Phi_\mu(\tau, x(\tau)) + \frac{\mu}{2} S \subset \Phi_{\mu/2}(\tau, x(\tau)).$$

Since $\Phi_{\mu/2}(\tau, x(\tau))$ is open, there are points $q_1, q_2 \in \partial\Phi_{\mu/2}(\tau, x(\tau)) = G_{\mu/2}(\tau, x(\tau))$ such that $q/m(J_{\tau,\lambda}) = \alpha q_1 + (1-\alpha) q_2$, for some $0<\alpha<1$. Hence, by a suitable partition of $J_{\tau,\lambda}$ in two intervals J_1 and J_2, we have $q = q_1 m(J_1) + q_2 m(J_2)$. Set, now, $\omega_{\tau,\lambda}(s) = q_1 \chi_{J_1}(s) + q_2 \chi_{J_2}(s)$, $s \in J_{\tau,\lambda}$, where χ_{J_i} denotes the characteristic function of J_i, $i=1, 2$. Observe that $\omega_{\tau,\lambda}$ is a measurable function with values $\omega_{\tau,\lambda}(s) \in G_{\mu/2}(\tau, x(\tau))$ and satisfies

$$\int_{J_{\tau,\lambda}} \omega_{\tau,\lambda}(s)ds = \int_{J_{\tau,\lambda}} \dot{x}(s)ds.$$

Define

$$z_{\tau,\lambda}(t) = x(\tau-\lambda) + \int_{\tau-\lambda}^{t} \omega_{\tau,\lambda}(s)ds, \qquad t \in J_{\tau,\lambda}.$$

Clearly, $z_{\tau,\lambda}(\tau \pm \lambda) = x(\tau \pm \lambda)$. Furthermore, for each $t \in J_{\tau,\lambda}$, we have

$$|z_{\tau,\lambda}(t) - x(t)| \leq \int_{J_{\tau,\lambda}} |\omega_{\tau,\lambda}(s) - \dot{x}(s)|\, ds < 2Mm(J_{\tau,\lambda}) < \frac{\delta(\tau)}{2} < \varepsilon.$$

Since $|z_{\tau,\lambda}(t) - x(\tau)| \leq |z_{\tau,\lambda}(t) - x(t)| + |x(t) - x(\tau)| < \delta(\tau)$, it follows that $(t, z_{\tau,\lambda}(t)) \in S((\tau, x(\tau)), \delta(\tau))$ and so, by (4.4),

$$\dot{z}_{\tau,\lambda}(t) = \omega_{\tau,\lambda}(t) \in G_{\mu/2}(\tau, x(\tau)) \subset F_{\mu}(t, z_{\tau,\lambda}(t)) \qquad \text{a.e. in } J_{\tau,\lambda}.$$

This completes the proof.

Lemma 4.2. *Let the hypotheses of Lemma* 4.1 *be satisfied. Then, there is a solution* $z\colon [t_0, T] \to X$ *of* (1.1) *such that*

$$|z(t) - x(t)| < \varepsilon \qquad \textit{for each } t \in [t_0, T], \tag{4.5}$$

$$\dot{z}(t) \in F_{\mu}(t, z(t)) \qquad \textit{a.e. in } [t_0, T]. \tag{4.6}$$

Proof. Let Δ^* be the set of the points of density of Δ_x^{μ}. It is well known that $m(\Delta^*) = m(\Delta_x^{\mu})$. If $m(\Delta^*) = 0$ there is nothing to prove. So let $m(\Delta^*) > 0$ and let $\tau \in \Delta^*$, $t_0 < \tau < T$. By Lemma 4.1 there is $\lambda_0(\tau) > 0$ such that for each $0 < \lambda < \lambda_0(\tau)$ there is a Lipschitzean function $z_{\tau,\lambda}\colon J_{\tau,\lambda} \to X$ which is differentiable a.e. and satisfies (4.1), (4.2), (4.3). Likewise in [2], consider the family of all closed intervals $J_{\tau,\lambda}$ where $\tau \in \Delta^*$, $t_0 < \tau < T$, and $0 < \lambda < \lambda_0(\tau)$. Since the intervals $J_{\tau,\lambda}$ are a Vitali's covering of Δ^*, by Vitali's theorem there is a countable subcovering of Δ^* by pairwise disjoint intervals $J_i = J_{\tau i,\lambda i}$ such that $m(\Delta^* \backslash \bigcup_i J_i) = 0$.

Set

$$\omega(t) = \sum_i \dot{z}_{\tau i,\lambda i}(t)\chi_{J_i}(t) + \dot{x}(t)\chi_{[t_0,T]\backslash\cup_i J_i}(t), \quad t \in [t_0, T] \quad \text{a.e.},$$

and define

$$z(t) = x_0 + \int_{t_0}^{t} \omega(s)ds, \qquad t \in [t_0, T].$$

Evidently z is Lipschitzean and $\dot{z}(t) = \omega(t)$ a.e.. Moreover z and x are equal at the end points of every interval J_i and at each $t \in [t_0, T] \backslash \bigcup_i J_i$. We prove only the first statement (the proof of the second is similar). To this end, set $J_i = [a_i, b_i]$ and denote by $\bigcup_k J_k$ the union of all intervals J_k (of the Vitali's subcovering of Δ^*) which are contained in $[t_0, a_i]$. We have

$$z(a_i)=x_0+\sum_k \int_{J_k} \omega(s)ds+\int_{[t_0,a_i]\setminus \cup_k J_k} \omega(s)ds$$
$$=x_0+\sum_k \int_{J_k} \dot{z}_{\tau_k,\lambda_k}(s)ds+\int_{[t_0,a_i]\setminus \cup_k J_k} \dot{x}(s)ds.$$

Thus

$$z(a_i)=x_0+\sum_k \int_{J_k} \dot{x}(s)ds+\int_{[t_0,a_i]\setminus \cup_k J_k} \dot{x}(s)ds=x(a_i)$$

and since, clearly, $z(b_i)=x(b_i)$ the statement is proved. Therefore

$$z(t)=\begin{cases} z_{\tau_i,\lambda_i}(t), & t \in J_i, \quad i=1, 2, \cdots \\ x(t), & t \in [t_0, T]\setminus \bigcup_i J_i \end{cases}$$

and so, by (4.2), we obtain (4.5). Furthermore,

$$\dot{z}(t)=\begin{cases} \dot{z}_{\tau_i,\lambda_i}(t), & t \in J_i \quad \text{a.e.}, \quad i=1, 2, \cdots \\ \dot{x}(t), & t \in [t_0, T]\setminus \bigcup_i J_i \quad \text{a.e..} \end{cases}$$

Since $m(\Delta^*\setminus \bigcup_i J_i)=0$ and the set $\Delta^* \subset \Delta_x^\mu$ satisfies $m(\Delta_x^\mu \setminus \Delta^*)=0$, there is a set $\hat{J} \subset [t_0, T]$ of measure zero such that $\Delta_x^\mu \subset \hat{J} \cup (\bigcup_i J_i)$. Thus, for almost all $t \in [t_0, T]\setminus \bigcup_i J_i$, we have $t \notin \Delta_x^\mu$, hence $\dot{x}(t) \in F_\mu(t, x(t))$. On the other hand, for almost all $t \in J_i$ we have $\dot{z}(t)=\dot{z}_{\tau_i,\lambda_i}(t) \in F_\mu(t, z_{\tau_i,\lambda_i}(t))=F_\mu(t, z(t))$. Therefore z satisfies (4.6) and the lemma is proved.

Now we are ready to prove that $\mathcal{N}_\theta$ is dense in $\mathcal{M}_F$.

Proof of Proposition 2.2 (*$\mathcal{N}_\theta$ is dense*). Let $x \in \mathcal{M}_F$ and fix $\varepsilon>0$. By Lemma 4.2 there is $z \in \mathcal{M}_F$ satisfying (4.5) and (4.6). By (4.6), $d(\dot{z}(t), \partial F(t, z(t))\leq \mu$ a.e. and so $z \in \mathcal{N}_\theta$ provided $\mu<\theta/(T-t_0)$. Since $|z(t)-x(t)|<\varepsilon$ for each $t \in [t_0, T]$, the set $\mathcal{N}_\theta$ is dense in $\mathcal{M}_F$.

§ 5. Proof of Proposition 2.2 ($\mathcal{N}_\theta$ is open).

In this section we prove the second statement of Proposition 2.2, namely that (if X is reflexive) the set $\mathcal{N}_\theta$ is open in $\mathcal{M}_F$.

By a simple application of Lebesgue's covering lemma it is easy to prove the following

Lemma 5.1. *Let F satisfy* (*). *Let K be a compact subset of Ω_1. Let $\varepsilon>0$. Then there is $\delta>0$ ($\delta<\min\{a, R\}$) such that for each $(t, u) \in K$ and all $(s, v) \in S((t, u), \delta)$ we have*

$$F(s, v)\subset F(t, u)+\varepsilon S.$$

Lemma 5.2. *Let $A \in \mathcal{B}$. If $u_1, u_2, \cdots, u_n \in A$ and $\sum_{i=1}^n \alpha_i=1$, $\alpha_i\geq 0$, then*

$$d\left(\sum_{i=1}^{n} \alpha_i u_i, \partial A\right) \geq \sum_{i=1}^{n} \alpha_i d(u_i, \partial A).$$

Proof. We observe that, for any $v \in A$, $d(v, \partial A) = \sup\{\beta \geq 0 \mid v + \beta S \subset A\}$. Let $\varepsilon > 0$. For each $i = 1, 2, \cdots, n$, there is $\beta_i > d(u_i, \partial A) - \varepsilon$, $\beta_i \geq 0$, such that $u_i + \beta_i S \subset A$. Since A is convex

$$A \supset \sum_{i=1}^{n} \alpha_i (u_i + \beta_i S) = \sum_{i=1}^{n} \alpha_i u_i + \left(\sum_{i=1}^{n} \alpha_i \beta_i\right) S,$$

which implies

$$d\left(\sum_{i=1}^{n} \alpha_i u_i, \partial A\right) \geq \sum_{i=1}^{n} \alpha_i \beta_i > \sum_{i=1}^{n} \alpha_i d(u_i, \partial A) - \varepsilon.$$

Since ε is arbitrary, the lemma is proved.

Proof of Proposition 2.2 (*$\mathcal{N}_\theta$ is open*). It is enough to prove that the set

$$\tilde{\mathcal{N}}_\theta = \left\{x \in \mathcal{M}_F \,\middle|\, \int_{t_0}^{T} d(\dot{x}(t), \partial F(t, x(t)))dt \geq \theta\right\}$$

is closed in $\mathcal{M}_F$.

Indeed, suppose that $\{x_n\} \subset \tilde{\mathcal{N}}_\theta$ converges uniformly to $x \in \mathcal{M}_F$. Let $\varepsilon > 0$. Set $K = \{(t, x(t)) \mid t \in [t_0, T]\}$ and let δ correspond (to ε and K) according to Lemma 5.1. There is n_0 such that for each $m \geq n_0$ and all $t \in [t_0, T]$ we have $(t, x_m(t)) \in S((t, x(t)), \delta) \subset \Omega_2$. Hence, by Lemma 5.1,

$$\dot{x}_m(t) \in F(t, x_m(t)) \subset F(t, x(t)) + \varepsilon S, \quad t \in [t_0, T], \quad m \geq n_0 \tag{5.1}$$

On the other hand, since X is reflexive and $\{x_n\}$ converges uniformly to x, by the argument of Proposition 2.1 it follows that a subsequence, say $\{\dot{x}_n\}$, converges weakly to $\dot{x}$ in $L^2([t_0, T], X)$. Hence a sequence of convex combinations $\{\sum_{i=0}^{k_n} \alpha_i^n \dot{x}_{n+i}\}$ converges strongly to $\dot{x}$ in $L^2([t_0, T], X)$ and so, in particular, in $L^1([t_0, T], X)$.

We have

$$\begin{aligned}\int_{t_0}^{T} d(\dot{x}(t), \partial F(t, x(t)))dt &\geq \int_{t_0}^{T} d(\dot{x}(t), \partial[F(t, x(t)) + \varepsilon S])dt - \varepsilon(T - t_0)\\ &\geq \int_{t_0}^{T} d\left(\sum_{i=0}^{k_n} \alpha_i^n \dot{x}_{n+i}(t), \partial[F(t, x(t)) + \varepsilon S]\right)dt\\ &\quad - \int_{t_0}^{T} \left|\sum_{i=0}^{k_n} \alpha_i^n \dot{x}_{n+i}(t) - \dot{x}(t)\right| dt - \varepsilon(T - t_0).\end{aligned}$$

Let $n \geq n_0$. Then, by virtue of (5.1) and Lemma 5.2, we have

$$\int_{t_0}^{T} d\left(\sum_{i=0}^{k_n} \alpha_i^n \dot{x}_{n+i}(t), \partial[F(t, x(t))+\varepsilon S]\right)dt$$
$$\geq \sum_{i=0}^{k_n} \alpha_i^n \int_{t_0}^{T} d(\dot{x}_{n+i}(t), \partial[F(t, x(t))+\varepsilon S])dt$$
$$\geq \sum_{i=0}^{k_n} \alpha_i^n \int_{t_0}^{T} d(\dot{x}_{n+i}(t), \partial F(t, x_{n+i}(t)))dt.$$

Therefore, since $x_{n+i} \in \tilde{\mathcal{N}}_\theta$, we obtain

$$\int_{t_0}^{T} d(\dot{x}(t), \partial F(t,x(t)))dt \geq \theta - \int_{t_0}^{T} \left| \sum_{i=0}^{k_n} \alpha_i^n \dot{x}_{n+i}(t) - \dot{x}(t) \right| dt - \varepsilon(T-t_0).$$

Let $n \to +\infty$. Since $\{\sum_{i=0}^{k_n} \alpha_i^n \dot{x}_{n+i}\}$ converges to $\dot{x}$ in $L^1([t_0, T], X)$ and ε is arbitrary, it follows that $x \in \tilde{\mathcal{N}}_\theta$. Thus $\tilde{\mathcal{N}}_\theta$ is closed, hence $\mathcal{N}_\theta$ is open and the proof of Proposition 2.2 is complete.

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